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Spinors and Multivectors as a Unified Tool for Spacetime Geometry and for Elementary Particle Physics

Jaime Keller[1]

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From the definition of spinors as the minimal left (right) modules of multivectors (that is, of vectors and their outer products), we can construct a unified mathematical approach for the study of matter and its interaction fields, which are either defined as fields in the geometrical spacetime or considered as generators of the physical spacetime. It is also shown how matter and interaction fields can be represented either by spinor fields or by multivector fields, both types of fields carrying the same information as the traditional corresponding spinors, vectors, or tensors. Geometry is more transparent in one representation (multivector form), and physics is more obvious in the spinor representation. Our theory provides a unified and totally self-consistent representation of quarks (barions), leptons, and all their known interactions.

1. INTRODUCTION

A standard approach for the construction of relativistic quantum theory is the definition of matter and interaction fields as sets of simple or composite spinor, vector, or higher-order tensor fields, which carry some "internal quantum numbers," in each point of a reference (flat or curved) spacetime. The reference frame called spacetime (usually taken as given) is given a geometrical structure needed to allow it to have a set of basic properties related to the "existence" in the real world of series of events and as a carrier of the trajectories of physical signals.

The basic signal trajectories for the definition of spacetime are those of light rays, according to the procedure started by Einstein when he used null vectors for the light trajectories as a basic concept in the definition of

[1]División de Estudios de Posgrado, Fac. de Quím., Universidad Nacional Autónoma de México, 04510 México, D.F. 70-528 México and Quantum Theory of Matter Project, Facultad de Estudios Superiores-Cuautitlán, Universidad Nacional Autónoma de México, 04510 México, D.F. 70-528 México.

spacetime, with the Lorentz transformations becoming a basic geometrical property of spacetime. In this way many laws of physics were formally transformed into geometrical relations.

In a sense it was unfortunate that, historically, spinor fields (and the quantization of action and angular momentum) could not be incorporated in these early definitions of the physical spacetime. They had to be added, in the formulation of the relativistic quantum field theories, as "new" types of covariant fields. Quantum mechanics incorporates the concept of the de Broglie phase of the different fields in such a way that the boundary conditions for the de Broglie phase induce the concepts of quanta of energy, momentum, and angular momentum. The de Broglie phase has remained as a mysterious "angle," at least in the sense that it has not been directly related to geometrical quantities or properties of spacetime. This situation can be somewhat annoying due to the discreteness (or "quantization") of angular momentum in units of $h/2\pi$, because angular momentum is definitely related to rotations in a given spacelike plane (of spacetime). Another indication of a possible geometrical meaning of h is the appearance of spin angular momentum, including half-odd-integer spins when the introduction of spinor fields in physics made it clear that the half-odd-integer representation of the Lorentz group belongs to the natural structure of spacetime.

In this paper we will show that the consideration of spinors and multivectors in a unified form clarifies all these physically related facts and allows a geometrical definition of the de Broglie phase, providing a unified formulation of the quantization of action and angular momentum, a geometrical interpretation of isotopic spin, weak charge, and color, as well as of the related interaction gauge fields, all this within the framework of the (multi)vectorial formulation of spacetime. A geometrical meaning of the appearance of rest mass when Riemannian spacetimes are considered is also included. As a corollary, we will show, first, that a collection of multivector fields, related to a matter field, carry the same information as the corresponding spinor fields, and, second, that the collection of vector (gauge) interaction fields can be considered as operators among the multivectorial or spinorial matter fields (therefore as multivector fields representing the interaction between the matter fields).

We will then be using two interchangeable points of view: (a) Given the geometry assumed for the physical spacetime, the observed matter and interaction fields are those that can exist in it. (b) given the known matter and interaction fields, they generate a framework for observation and description which is the (multi)vector spacetime.

As a result of this tautomery, we will show that the physical spacetime can be considered as generated by the collection of interacting multivector matter fields in the process of defining a "free" particle, an abstraction which is only possible when an observer constructs a theory for physics within the frame of what he defines as his physical spacetime.

The mathematical and physical theory here presented corresponds, in the appropriate limits, to the standard theories of elementary particles and their interaction fields, in all their aspects, and provides a natural link with general relativity.

The paper will be developed in a self-consistent way, providing the definitions of all quantities used. An attempt is made to use the standard terminology whenever no misinterpretation can result. Even when new concepts are introduced, they are given the equivalent accepted names when the identification with more widely accepted terms is possible. The multivector and spinor algebra is fully included to facilitate the reading of the paper.

2. THE MATHEMATICAL SPINOR-VECTOR-MULTIVECTOR STRUCTURE OF COMPLEX SPACETIME

2.1. Some General Considerations

In this section we present in a systematic notation and in a unified representation the structure of spacetime as a system of vectors, multivectors, and spinors. There are several reasons for condensing here this mathematical system, besides the definition of our notation, the most relevant one being the explicit formulation of the geometry of spacetime in terms of spinors and their "outer" products, which is the fundamental tool that will allow us to show the geometrical meaning of many quantities in elementary particle physics. Moreover, some relationships have not been discussed before and we need a correct formal definition of the quantities involved. The use in Section 4 of Riemannian geometry also requires the analysis of the covariant derivatives of (multi)vectors and of spinors. The notation is a development of the one used in previous papers (Keller, 1986*a*,*b*).

We must start by considering the physical spacetime $\mathcal{M}$ as an orthogonal space $\mathcal{M} = R^{1,3}$ over R with a nondegenerate quadratic form

$$f(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{1}$$

and an associative algebra (containing $R^{p,q}$, $N = p + q$, as a subspace), where the square of a "vector" x equals its quadratic form $x^2 = f(x)$, which is the Clifford algebra $R_{1,3}$ with a ring of dimension $2^N = 16$. The Clifford algebra is the direct sum of the multivector spaces, each k-vector space is of dimension $\binom{N}{k}$ with $0 \leq k \leq N$, and, of course, $\sum_k \binom{N}{k} = 2^N$. A basis k-vector is given by

$$e_A(k) = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}, \qquad 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq N \tag{2}$$

The treatment given here could be made very general, for any orthogonal space $R^{p,q}$ (see, for example, Lounesto, 1980), but in fact most applications to physics of vector Clifford algebras involve spinors and multivectors in spaces with five dimensions or less, which are the cases discussed here explicitly. Historically, a Clifford algebra of two-dimensional space was introduced at the beginning of the last century by Wessel and thoroughly discussed by Hamilton in his famous formula $i^2 = j^2 = k^2 = ijk = -1$ to study (surprisingly, because the duality between surfaces and vectors in R^3 was not known) three-dimensional rotations and, independently, by Pauli in the study of the spin algebra $\mathcal{P}$. The multivectors for three-dimensional space $\mathcal{P}_c$ were introduced, at the beginning of the century, either as complex quaternions by Hamilton or as the standard three-dimensional space vector algebra, through the definition of axial and polar vectors, by Gibbs. The cases of four-dimensional $\mathcal{D}$ and five-dimensional (complex four) $\mathcal{D}_c$ algebras were developed by Dirac, who also introduced the use of bispinors [the Clifford algebras mentioned here are isomorphic to matrix algebras: $\mathcal{D}_c \sim C(4)$, $\mathcal{D} \sim H(2)$, $\mathcal{P}_c \sim C(2)$]. Because we only use in this paper $\mathcal{D}_c$ and their subspaces, our presentation is restricted to this particular algebra. The basic references for spinors remain Cartan (1913), Brauer and Weyl (1935), and Cartan's book of 1937, later reprinted (Cartan, 1981), which includes a thorough discussion of multivectors and of the mapping of spinors into multivectors, the so-called Cartan map.

We are also not interested here in the particular cases $p - q = 1 \bmod 4$ (where the Clifford algebra generated by the totally antisymmetric products of the orthonormal basis of $R^{p,q}$ can be decomposed into two ideals and the representation requires special treatment), except as subspaces of $\mathcal{D}_c$, which will automatically be included in our presentation. We will not be omitting then any fundamental case in the formalism used here.

Emphasis will be put here on the vector, multivector, and spinor basis sets and their relationships, which will be denoted by e_α (vector basis), e_A (multivector basis), χ_a (spinor basis), and $\chi_b^\dagger$ (conjugated spinor basis). Many equations will be written relating the different basis sets, for example, definitions like $e_B = e_\alpha e_\beta$, where putting together e_α and e_β means multiplication of those two basis elements, or in the same fashion $e_D\chi_d$, $\chi_f^\dagger e_G$, $\chi_d\chi_f^\dagger$, and $\chi_f^\dagger\chi_d$ represent multiplication of a multivector onto a spinor, (conjugated) spinor into a multivector, spinor into a (conjugated) spinor, and (conjugated) spinor into a spinor, respectively; if a representation is made, the *same* multiplication rule should be used in all cases. It is well known that matrix multiplication is an appropriate representation, and that the e_A

are represented by square matrices and that the χ_a ($\chi_b^\dagger$) are represented by column (row) matrices with matching dimensions.

The use of χ_a and $\chi_b^\dagger$ spinors (usually called column or row spinors because of the foreseen representation) enlarges the multivector Clifford algebra generated by the e_A into a new mathematical system $K_N \supset R_{p,q}$. We are aware that several possible formalisms for K_N are useful, but only one will be used in this paper, which is better adapted to the uses and notation of elementary particle physics. We will consider Dirac spinors (4 entries) as the fundamental mathematical entities and consider Weyl spinors (2 entries) as a Dirac spinor where a restriction has been made. A spinor pair will then consist of two Dirac spinors (Weyl spinors if the restrictions apply), etc.

2.2. The *K* Algebra Generated by the Spinors

Consider the primitive concepts of the *spinor* space $\mathfrak{L}$ with spinor basis χ_a and its *dual spinor* space $\mathfrak{L}^\dagger$ with spinor basis $\chi_a^\dagger$, provided with a symmetric spinor metric, given by the *spinor inner product*

$$\chi_b^\dagger \chi_a = c_{ab}; \qquad \chi_b^\dagger \in \mathfrak{L}^\dagger, \quad \chi_a \in \mathfrak{L} \tag{3}$$

c_{ab} is a complex number (in this paper $c_{ab} = \delta_{ab}$) and $a, b = 1, 2, \ldots, n$. They generate a set of $2n^2$ linearly independent *multivectors* $\mathcal{M} \in \mathcal{D}_c$ as the linear combination of the *spinor outer products*

$$\mathbb{M} = \sum_{a,b} \mathbb{M}^{ab} \chi_a \chi_b^\dagger \tag{4}$$

where the $\mathbb{M}^{ab}$ are complex coefficients. $\mathcal{D}_c$ represents the multivector space.

The set of multivectors and spinors has the closure property

$$\text{if} \quad \chi \in \mathfrak{L} \text{ and } M \in \mathcal{D}_c \quad \text{then } \mathbb{M}\chi \in \mathfrak{L} \tag{5}$$

and for the dual spinor space

$$\text{if} \quad \chi^\dagger \in \mathfrak{L}^\dagger \text{ and } \mathbb{M} \in \mathcal{D}_c \quad \text{then } \chi^\dagger \mathbb{M} \in \mathfrak{L}^\dagger \tag{6}$$

where $\chi = \sum_a \psi^a \chi_a$ and $\chi^\dagger = \sum_b \psi^{\dagger b} \chi_b^\dagger$, and ψ^a and $\psi^{\dagger b}$ are complex coefficients. The closure is proven by direct substitution of (3) and (4) into (5) or (6). The spinors are then modules of the multivectors, $\mathfrak{L}$ a left module and $\mathfrak{L}^+$ a right module. In the case $n = 2^p$ the number of multivectors $D =$ is $2n^2 = 2^{2p+1}$, and $2p = N$ is the *dimension of the vector space* generating a closed multivector algebra, discussed below, that is, the number of anticommuting elements needed to generate $\mathbb{M}$. If $n \neq 2^p$, a new type of algebra is generated, the discussion of which is beyond the scope of the present paper (Keller, 1988; Finkelstein *et al.*, 1986). If in (3) and (4) c_{ab} and $\mathbb{M}^{ab}$ are restricted to be real numbers, then the dimension of the multivector algebra is reduced to $D = n^2 = 2^N$. The dimension D corresponds to the number of degrees of freedom in the linear Clifford algebra which has been constructed.

Corollary. The multivectors $\mathbb{M} \in \mathfrak{D}_c$ constitute a (complex) associative algebra (of dimension $D = 2^{N+1}$).

Proof. If $\mathbb{M}$, $\mathbb{M}'$, $\mathbb{M}'' \in \mathfrak{D}_c$ and $d, f \in \mathbb{C}$, then:

(1) $\mathbb{M} + \mathbb{M}' = \mathbb{M}''$ by (4) (7a)

(2) $d(\mathbb{M} + \mathbb{M}') = d\mathbb{M} + d\mathbb{M}'$ by (4) (7b)

(3) $\mathbb{M}\mathbb{M}' = \mathbb{M}''$ by (3) and (4) (7c)

(4) $\mathbb{M}(\mathbb{M}'\mathbb{M}'') = (\mathbb{M}\mathbb{M}')\mathbb{M}''$ by (3) and (4) (7d)

(5) If the particular representation is chosen where $c_{ab} = \delta_{ab}$, then the multiplicative unit element can be written $1 = \sum_a \chi_a \chi_a^\dagger$ and it is immediate that

$$1\mathbb{M} = \mathbb{M}1 = \mathbb{M} \tag{7e}$$

and 0 is the addition unit element.

We can moreover define a set of 2^{n+1} *basis multivectors* $\mathbb{M}_A$ (or k-vectors) such that any multivector $\mathbb{M} = \sum_A m^A \mathbb{M}_A$ and $\mathbb{M}_A \mathbb{M}_A = \pm 1$, to generate a normed ring $\mathbb{M}_A \mathbb{M}_B = \mathbb{M}_c$ because $(\mathbb{M}_A \mathbb{M}_B) = (\mathbb{M}_c)^2 = \pm 1$. Also, N (or, for a *real* algebra, $N+1$) mutually anticommuting elements $\mathbb{M}_u$ of the ring can be chosen to generate the entire ring. The $e_u = \mathbb{M}_u$ elements are called the *basis vectors* of the ring. Any basis k-vector is constructed then as the product of k different basis vectors as defined below.

Then the spinors generate the multivectors and the spinors in turn are the modules of the multivectors. The set of spinors and multivectors together generate a closed mathematical system we have called K_N, defined by (3) and (4). In the following section we will use the system itself to generate the (complex) Riemannian $K_4^{(c)}$ spacetime and study its geometrical properties.

For K_q we will use either (a) the four entries given by Dirac spinors $\chi \in \mathfrak{L}_{2,2}$ or bispinors [consistent with a Weyl spinor $\zeta_{1b} \in \mathfrak{L}_{2,0}$ and a conjugated Weyl spinor $\zeta_{2b} = \zeta_{1b}^c \in \mathfrak{L}_{0,2}$; then $\chi_b^\dagger = (\zeta_{1b}^\dagger, \zeta_{2b}^\dagger)$, the c above a spinor denoting spinor conjugation, $\zeta^c = \varepsilon\zeta^*$, where ε is the antisymmetric symbol (matrix) for two dimensions and the asterisk denotes complex conjugation of the coefficients] or (b) a pair ω_b of Weyl spinors, $\omega_b^\dagger = (\zeta_{1b}^\dagger, \zeta_{2b}^\dagger) \in \mathfrak{L}_4^\dagger$, which can be mapped onto the Dirac spinors $\chi^\dagger \in \mathfrak{L}_{2,2}^\dagger$.

In this second case, when a spinor pair (both of the same representation) is used, the multivector representation of the improper orthogonal transformations is not possible. In every spinor dimensionality $n = 2^p$ the distinction between bispinors and a spinor pair is possible and meaningful, the component spinors being of dimension $m = 2^{p-1}$.

2.3. Dirac Spinors and the Multivectors Generated by Their Outer Products

The vectorial space $\mathfrak{L}_{2,2}$ of the Dirac spinors with basis elements $\chi_a \in \mathfrak{L}$, $a = 1, 2, 3, 4$, and its dual space $\mathfrak{L}^\dagger_{2,2}$ with basis elements $\chi^\dagger_a \in \mathfrak{L}^\dagger_{2,2}$ are taken as the primitive set of elements. A general element is $\psi = \sum_a \psi^a \chi_a \in \mathfrak{L}_{2,2}$ or $\psi^\dagger = \sum_a \psi^{a\dagger} \chi^\dagger_a \in \mathfrak{L}^\dagger_{2,2}$.

A spinor metric is defined by the (noncommutative) product

$$\chi^\dagger_a \chi_b = \delta^a_b \tag{8a}$$

and a "multivector" ϑ^b_a by the (noncommutative) product

$$\vartheta^b_a = \chi_a \chi^\dagger_b \tag{8b}$$

We have already indicated that a proper representation will require that (6) and (8) are automatically satisfied; the representation of χ_a by a column matrix and $\chi^\dagger_b$ by the transpose of the complex conjugate of the column matrix, with four entries each, constitutes a faithful representation. The products[2] are then, simply, matrix products and the full algebra of the K-system a matrix algebra. A representation of the equivalent system for Euclidean 3-space can be found in Hamilton (1984).

There are 16 elements ϑ^b_a, which have a direct meaning as operators in spinor space from (6) and (8). As each of them acts nontrivially only on one spinor χ_b (or $\chi^\dagger_b$ on the right), it is more meaningful to construct linear combinations which have a nontrivial action on the complete set of four χ_b (or $\chi^\dagger_a$), mapping each and all spinors χ_b in a corresponding spinor χ_a:

$$\mathbb{M}_A = A_1 \chi_{a_1} \chi^\dagger_1 + A_2 \chi_{a_2} \chi^\dagger_2 + A_3 \chi_{a_3} \chi^\dagger_3 + A_4 \chi_{a_4} \chi^\dagger_4 \tag{9}$$

with the particular, normalized choice of the complex coefficients

$$A_a \neq 0 \qquad \text{and} \qquad \sum_a |A_a|^2 = 4 \tag{10}$$

the set $\{\mathbb{M}_A\}$ constitutes a Clifford ring (of 16 or 32 elements in our case) and generates a Grassmann–Clifford algebra (invariant under a similitude transformation).

[2]There is in fact only one type of product in the algebra of the χ_a, $\chi^\dagger_b$, and the ϑ^b_a; this product can be appropriately represented by matrix multiplication. The outer product $\vartheta^b_a = \chi_a \chi^\dagger_b$ results in ϑ^b_a being a square matrix. That is, if the matrix elements of χ_c and ϑ^b_a are $(\chi_c)_n$ and $(\vartheta^b_a)_{mn}$, respectively $(m, n = 1, 2, 3, 4)$, then $\vartheta^b_a \chi_c = \chi_a \delta^b_c = \sum_n (\vartheta^b_a)_{mn} (\chi_c)_n = (\chi_a)_m \delta^b_c$ as far as $(\chi^\dagger_b)_{1n} (\chi_c)_{n1} = (\delta^b_c)_{11}$.

For example, the *identity* operator or 1 (or scalar unit) is

$$1 \equiv \chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger + \chi_3\chi_3^\dagger + \chi_4\chi_4^\dagger = +\vartheta_1^1 + \vartheta_2^2 + \vartheta_3^3 + \vartheta_4^4 \tag{11}$$

the *chirality* operator $\mathbb{M}_{ch}$ (up to a phase $e^{i(ch)}$)

$$\mathbb{M}_{ch} = -\chi_1\chi_1^\dagger - \chi_2\chi_2^\dagger + \chi_3\chi_3^\dagger + \chi_4\chi_4^\dagger = -\vartheta_1^1 - \vartheta_2^2 + \vartheta_3^3 + \vartheta_4^4 \tag{12}$$

and the *spin* operator $\mathbb{M}_s$ (up to a phase $e^{i(s)}$) is

$$\mathbb{M}_s = \chi_1\chi_1^\dagger - \chi_2\chi_2^\dagger + \chi_3\chi_3^\dagger - \chi_4\chi_4^\dagger = +\vartheta_1^1 - \vartheta_2^2 + \vartheta_3^3 - \vartheta_4^4 \tag{13}$$

All of them map $\chi_a \to \pm\chi_a$, and are the basic operators to classify the spinors according to a pair of attributes. $\mathbb{M}_{ch}$ is usually called γ_5.

There should be no physical implication in calling them the *chirality* and the *spin*, but they are given here the names that will allow the immediate identification of the χ_a as a basis set for spinors with physical significance. (For the case of Euclidean 3-space only one attribute is needed, which is universally named *spin*. No widespread name has been given for attributes needed for dimensions higher than 5 or for spaces with 5 dimensions and signature $p+q=5$ and $p-q=1 \bmod 4$. The "standard" representation in quantum mechanics can be obtained from the "chiral" representation by a linear transformation and therefore there is no conflict in using the "chiral"-"spin" representation here.)

The three operators 1, $\mathbb{M}_{ch}$, and $\mathbb{M}_s$ can be combined in a number of important ways (here $P_\lambda^2 = P_\lambda$):

The spin projectors are

$$P_\uparrow = \tfrac{1}{2}(1+\mathbb{M}_s); \qquad P_\downarrow = \tfrac{1}{2}(1-\mathbb{M}_s) \tag{14a}$$

the chirality projectors are

$$P_L = \tfrac{1}{2}(1-\mathbb{M}_{ch}); \qquad P_R = \tfrac{1}{2}(1+\mathbb{M}_{ch}) \tag{14b}$$

and the spinor projectors are

$$\vartheta_1^1 = P_L P_\uparrow; \qquad \vartheta_2^2 = P_L P_\downarrow; \qquad \vartheta_3^3 = P_R P_\uparrow; \qquad \vartheta_4^4 = P_R P_\downarrow \tag{15}$$

which project, respectively, spin up, spin down, left-handedness, right-handedness, and the spinor basic elements of the set χ_a ($\chi_a^\dagger$ on the right).

The operator $\mathbb{M}_s\mathbb{M}_{ch} = \mathbb{M}_{ch}\mathbb{M}_s$ appears in the spin selector operators just defined.

Given the previous operators, there are four others that are especially important, the step operators that map ("transform") the set $\{\chi_a\}$ into itself: each $\chi_a \to \chi_{a'} \neq \chi_a$. We can choose this mapping in the basic four types (see below for an additional explanation of the choice presented here).

e_0. Change chirality, but not spin.

e_1. Change chirality and spin. No change in spinor relative complex phase.

e_2. Change chirality, spin, and the spinor relative phase by $e^{i\odot} = \pm i$. Change the phase of R to L by ± 1.

e_3. Change chirality, not the spin, changing the spinor relative complex phase by $e^{i\odot} = \pm 1$ and the R to L phase by ± 1.

It would be easy to show that any other possible mapping is realizable, using a combination of the e_α ($\alpha = 0, 1, 2, 3$) and $\mathbb{M}_{ch}$ (or $\mathbb{M}_s$). For example, $e_0 e_1$ will change the spin only, $e_0 e_3$ gives a relative phase only, etc.

The set e_α anticommutes with $\mathbb{M}_{ch}$ and anticommutes among itself, $e_\alpha \mathbb{M}_{ch} = -\mathbb{M}_{ch} e_\alpha$ and $e_\alpha e_\beta = -e_\beta e_\alpha$, $\beta \neq \alpha$, all $\alpha, \beta = 0, 1, 2, 3$. Also, $e_0^2 = 1$ and $e_i e_i = -1$ for $i = 1, 2, 3$.

The anticommuting extended five-member set $\{e_\alpha, \mathbb{M}_{ch}\}$ can generate the complete Clifford ring of operators on the Dirac spinors. This is then a basic ring for the algebra $R_{2,3} \approx R_{0,5} \approx R_{4,1}$ containing as subrings $R_{0,4}$, $R_{1,3}$, $R_{2,2}$, $R_{3,1}$, and $R_{4,0}$ and their complexifications [$R_{1,3}$ or $R_{3,1}$ are the spacetime geometric algebras of signature $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $g_{\mu\nu} = \mathrm{diag}(1, 1, 1, -1)$, respectively].

The explicit forms of e_α are (up to a complex phase factor)

$$e_0 = \vartheta_3^1 + \vartheta_4^2 + \vartheta_1^3 + \vartheta_2^4 \tag{16a}$$

$$e_1 = \vartheta_4^1 + \vartheta_3^2 - \vartheta_2^3 - \vartheta_1^4 \tag{16b}$$

$$e_2 = i(-\vartheta_4^1 + \vartheta_3^2 + \vartheta_2^3 - \vartheta_1^4) \tag{16c}$$

$$e_3 = \vartheta_3^1 - \vartheta_4^2 - \vartheta_1^3 + \vartheta_2^4 \tag{16d}$$

The particular choice of the four e_α was made to fulfill the condition

$$\mathbb{M}_{ch} = i e_0 e_1 e_2 e_3 \tag{17}$$

ensuring the condition $e_\alpha \mathbb{M}_{ch} = -\mathbb{M}_{ch} e_\alpha$ given above (the factor i makes $\mathbb{M}_{ch}$ linearly independent from $e_0 e_1 e_2 e_3$!). The set of e_μ will be used to give the spacetime its best known representation with signature $R^{1,3}$. Also,

$$\mathbb{M}_s = i e_2 e_3 \tag{18}$$

The defining properties of the e_α can be seen in a more apparent form by remembering that we have chosen

$$\begin{aligned} \mathbb{M}_{ch}\chi_1 &= -\chi_1 \quad \text{and} \quad \mathbb{M}_s\chi_1 = +\chi_1 \\ \mathbb{M}_{ch}\chi_2 &= -\chi_2 \quad \text{and} \quad \mathbb{M}_s\chi_2 = -\chi_2 \\ \mathbb{M}_{ch}\chi_3 &= +\chi_3 \quad \text{and} \quad \mathbb{M}_s\chi_3 = +\chi_3 \\ \mathbb{M}_{ch}\chi_4 &= +\chi_4 \quad \text{and} \quad \mathbb{M}_s\chi_4 = -\chi_4 \end{aligned} \tag{19}$$

In the case of e_0, using $e_0\mathbb{M}_{ch} = -\mathbb{M}_{ch}e_0$ and the fact that $R_{p,q}$ is an associative algebra, we have

$$\mathbb{M}(e_0\chi_{a'}) = -e_0(\mathbb{M}\chi_{a'}) = \begin{cases} (e_0\chi_{a'}), & a'=1,2 \\ -(e_0\chi'_a), & a'=3,4 \end{cases} \tag{20}$$

and therefore e_0 has made a mapping $\chi_{a'} \to e_0\chi_{a'} = \chi_a$ which has changed the chirality of $\chi_{a'}$. A projector can be defined $P_+ = (1+e_0)/2$.

The definitions of the e_i, $i = 1, 2, 3$, are also shown to be fulfilled in the same way.

The anticommuting set e_α generates a vector space $R^{1,3}$ and a Clifford ring $R_{1,3}$; they are therefore called a *vector* basis set of $R^{1,3}$, a local space with the geometric properties of spacetime.

The fact that the basis vectors we have chosen (or an odd product of vectors), when acting on a spinor, "change" the chirality of every elementary spinor (that is, of a spinor corresponding to a definite handness and spin) is very important. Vector (or trivector $e_\alpha e_\beta e_\gamma$, $\alpha \neq \beta \neq \gamma \neq \alpha$) operations on definite spin-chirality spinors will "change" them, but composite spinors, of the type $A^L\chi_L + A^R\chi_R$, can be mapped into themselves. In the construction of the wave equations for spinors (below) and the study of their gauge invariance properties the mapping of left-handedness into right-handedness and vice versa will introduce an additional term in the equations which breaks the chiral symmetry and generate the rest mass of the particles.

To return to the vector and multivector space generated by the spinors χ_a through their outer products, there are 16 elements ϑ_a^b which, when combined through complex coefficients, generate a ring of 32 linearly independent objects (the isomorphic multivector Clifford algebras $R_{0,5} \approx R_{2,3} \approx R_{4,1}$) containing a set of five mutually anticommuting elements (for the case of $R_{2,3}$ they can be chosen to be $\mathbb{M}_{ch}$, e_0, e_1, e_2, and e_3 as above). For the purpose of systematization we can, however, make other choices which will have formal advantages.

I. Consider the Clifford algebra to be $R_{0,5}$ with basis vectors $\mathbf{e}_v$ ($v = 1, 2, 3, 4, 5) \in R^{0,5}$ defined in terms of the products of our previously given $R^{1,3}$ vectors e_α as

$$\mathbf{e}_v = \{ie_1e_2e_3,\ ie_0,\ ie_0e_1,\ ie_0e_2,\ ie_0e_3\} \tag{21}$$

Then the e_v generate a 32-element Clifford ring $\mathscr{D}_c = R_{0,5}$,

$$\mathscr{D}_c \equiv \{1, \mathbf{e}_v, \mathbf{e}_{vu}, \mathbf{e}_{muv}, \mathbf{e}_{mnuv}, i\} \tag{22}$$

where $\mathbf{e}_{uv} \cdots = \mathbf{e}_u\mathbf{e}_v \cdots$ is a multivector.

The *even* part of $\mathscr{D}_c$, denoted $\mathscr{D}_c^+$, that is, the set of multivectors with an even number of products of the e_u elements, is

$$\mathscr{D} = R_{1,3} = \{1, \mathbf{e}_{uv}, \mathbf{e}_{mnuv}\} = \{1, e_\alpha, e_{\alpha\beta}, e_{\alpha\beta\gamma}, e_{\alpha\beta\gamma\delta}\} = \mathscr{D}_c^+ \tag{23}$$

the real spacetime algebra containing 16 elements (here $\alpha = 0, 1, 2, 3$ as above) represented by the Dirac γ_μ matrices.

The even part of $\mathscr{D}$, in turn, generates the Clifford ring P_c with 8 elements (here $i = 1, 2, 3$)

$$P_c = R_3 = \{1, e_{\alpha\beta}, e_{\alpha\beta\gamma\delta}\} = \{1, e_i, e_{ij}, e_{ijk}\} = \mathscr{D}^+ \tag{24}$$

the algebra of Euclidean 3-space.

The algebras generated by the Clifford rings given above are called: $\mathscr{D}_c$ = Dirac complex (or complex spacetime), $\mathscr{D}$ = Dirac, P_c = Pauli complex. Finally, the even part of P_c is

$$P = \{1, e_{ij}\} = \{1, i, j, k\} = \mathbb{H} \tag{25}$$

and corresponds to the quaternion algebras $\mathbb{H}$ [here called the Pauli algebra, because it is generated by the Pauli matrices σ_i' with norm $(\sigma_i')^2 = 1$].

The fact that each successive algebra is the even part of the previous one generates a reduction chain of even subalgebras

$$\mathscr{D}_c \to \mathscr{D} \to P_c \to P \to C^1 \to R^1 \tag{26}$$

with basis vector spaces of $N = 5, 4, 3, 2, 1, 0$ dimensions, respectively. Also, $\mathscr{D}_c = \mathscr{D} + i\mathscr{D}$ and the 32 elements of $\mathscr{D}_c$ can, faithfully, be decomposed into 16 *real* and 16 *imaginary* multivectors.

II. Another important reduction chain that can be used for physical analysis is one in which the basis vectors of each subalgebra (dimension $N-1$) is a *cut* of the basis vectors of the previous one (dimension N) by multiplication by one of the basis vectors of the N-dimensional algebra. As an example, consider as starting point the basis vectors

$$f_u = \{\mathbb{M}_{ch}, \mathbb{M}_{ch}e_1, \mathbb{M}_{ch}e_2, \mathbb{M}_{ch}e_3, \mathbb{M}_{ch}e_0\} \tag{27}$$

(here $u = 1, 2, 3, 4, 5$) generating the Clifford ring $R_{4,1} \approx \mathscr{D}_c$.

The *cut* $\mathbb{M}_{ch}f_u = \{1, e_1, e_2, e_3, e_0\}$ produces the basis for the ring $R_{1,3}$ and the second *cut* (or spacetime cut)

$$e_0 e_\alpha = \{1, e_{01}, e_{02}, e_{03}\} \tag{28}$$

produces a basis for the ring R_3 of Euclidean 3-space, isomorphic to the Pauli matrices, used in what is usually called spinor analysis where two-component spinors are considered.

The examples presented here should be enough to show that the geometries generated by particular linear combination of ϑ_a^b cover the more widely used approaches to spacetime algebras used in physics. Our emphasis has been on the vector and multivector basis set, rather than on the components of the vectors and multivectors; then the formulas we will obtain are representation-independent formulas. The fact that we have chosen a nomenclature (chirality and spin) to simplify the application in physics of the algebras generated is not restrictive insofar as they have not been given any physical meaning yet.

In Appendix A we present some relevant relations of the algebra, both for spinors and for multivectors. A comprehensive reference to Clifford algebras as used in physics is Chisholm and Common (1986).

3. SPACETIME GENERATED BY A LARGE COLLECTION OF INTERACTING SPINOR FIELDS

3.1. The Universe Considered as a Collection of Spinor Fields

In the previous section we have seen that even the consideration of a single spinor field ψ generates a mathematical structure through the (extended Cartan) mapping

$$\psi = \sum_a \psi^a \chi_a \rightarrow \mathbb{M} = \psi\psi^\dagger = \sum_{ab} \mathbb{M}^a_b \vartheta^b_a; \qquad \mathbb{M}^a_b = \psi^a \psi^\dagger_b \in C^1 \tag{29}$$

A collection of interacting spinor fields can be described either as a single but very complex unit, or as generating an average system, with some properties where every field can be described as almost "free" interacting in a particular way in the background of the average system. The properties of the fields are in any case redefined in terms of the background system to which it belongs. The resulting elementary spinor fields and their interaction fields will be either simple or composite.

The word field has been introduced from the beginning because the description in terms of the properties of the average system requires the use of a set of parameters $\{x_\rho\}$ to relate the spinor (and its interactions) to the system.

The system is called the universe, an appropriate name because for each interacting field the rest of what is relevant to it is included in the universe.

A series of postulated principles will be used, as follows.

Postulate I. Existence. The universe consists of a collection of interacting fields, and has the mathematical characteristics generated by the fields themselves and their interactions.

All fields in turn must be redefined in terms of the properties of the universe and must be such that they can exist in it, in the sense of mathematical compatibility. All fields that are compatible with the mathematical characteristics of the universe should also exist, or the universe must be redefined otherwise.

As a consequence, it is of no physical importance whether we start by postulating the existence of the universe and its average properties (usual procedure) or by postulating the existence of the interacting fields, the collection of which will generate the universe (the procedure that will be followed in this paper). The idea of constructing the universe from elementary notions related to fields has been proposed by several authors, for example, Marlow (1984) and Penrose (1971).

Many types of universes can then be thought of as being constructed from interacting fields, but we have chosen to start from Dirac spinor fields (with 2^p degrees of freedom, $p = 2$) because we want to construct the universe of the physical spacetime where matter, radiation, and the physical vacuum (the average universe), in the way we know, exists.[3] Composite fields will be considered to result from the combination of Dirac spinors.

In order to construct a meaningful average universe, or at least a universe where uncertainties are not so large that they could render useless the definition of its geometrical elements, we need a second postulate referring to the possibility of creating a system consisting of an average background and singling out of it fields (free or interacting) for study.

Postulate II. The universe consists of a very large collection of interacting fields and their relation is such that it can be considered to be isotropic and homogeneous, at least in a first approximation.

The homogeneous universe constructed in that way will acquire several properties:

(a) It is finite if the collection of interacting fields generating the physical frame is finite. The simplest form (to construct a vector space) is the use of the mapping of the fields into multivectors (we have shown in Section 2 that those multivectors belong to the Clifford algebra $R_{0,5}$ with vector basis e_u); then each field "in the universe" ψ_λ generates a set of multivectors which have to be related to the multivectors of the other fields $\psi_{\lambda'}$.

(b) As each field can be taken as a reference, then the others have to be referred to it in a particular way. We will show in Sections 4 and 5 that this is done through an adjustment of a local "phase."

A multivector in complex spacetime is a collection of scalars, vectors e_μ, planes, volumes, and hypervolumes; then a possible relation is to relate the k-vectors generated by ψ_λ to the k-vectors of $\psi_{\lambda'}$. As the collection of vectors generates the notion of position and distance, all the other terms

[3]We will discuss elsewhere the lack of dynamics if Weyl spinors, $p = 1$ (with $2^p = 2$ degrees of freedom), are strictly used as the basic matter fields and the great degeneracy if spinor fields with $2^p > 4$ degrees of freedom are used ($p > 2$).

correspond to derived notions. A most important notion is of course that of a volume in which the universe exists. The (finite) universe has a (finite) volume. Then each field is considered to have a probability amplitude per unit volume and the expression $\psi^\dagger e_0 \psi$ to correspond to a vector called the four-current density with components $(\psi^\dagger e_0 e_0 \psi, \psi^\dagger e_0 e_1 \psi, \psi^\dagger e_0 e_2 \psi, \psi^\dagger e_0 e_3 \psi)$. Distances are generated by the notion of interaction. The basic interactions, to be provisionally called "light," will be shown below to generate the basic notion of a distance as a world line $s^2 = (x_0)^2 - x_1^2 - x_2^2 - x_3^2 = 0$ and then the basic local metric.

The relation of one basic elementary matter field ϕ_λ to the background universe is given by considering that it is described by a nondecaying spinor field ψ_λ (in the abstraction of a "free" field) in the generated spacetime ST. This notion requires the mapping of ϕ_λ onto the *field in spacetime* ψ_λ such that

$$\phi_\lambda \to \psi_\lambda = \sum_a \psi_\lambda^a(x)\chi_a \tag{30}$$

where the basis spinor set χ_a generates the basis multivectors $\mathbb{M} = \sum_{a,b} \mathbb{M}^{ab}\chi_a\chi_b^\dagger$ and the spinor components ψ_λ^a contain both the information on the existence of the spinor in ST and the information on its interactions with the homogeneous and isotropic background universe, with the local deviations of ST from homogeneity and isotropy, and the interaction of ϕ_λ with other fields.

For the definition of the ψ_λ^a we need a form to relate ψ_λ to ST:

Postulate III. All angular momenta L_{uv} related to rotations of the matter or interaction field in the plane $e_u \wedge e_v$ of the generated mathematical space $R_{0,5}$ (with Clifford algebra $\mathscr{D}_c$) are quantized.

This quantization in the planes of $R_{0,5}$ corresponds in the first place to the well-known quantization $L_{ij} = n(h/2\pi)$ of angular momentum in ST, because from (21) the planes e_{34}, e_{45}, and e_{53} in $R_{0,5}$ correspond to e_{12}, e_{23}, and e_{31} in $R_{1,3}$, respectively.

There is also a second type of rotation S in $\mathscr{D}_c$, occurring in the (abstract in spacetime) plane

$$e_1 \wedge e_2 = (ie_1e_2e_3) \wedge (ie_0) = e_0e_1e_2e_3 = \gamma_5 \tag{31a}$$

which corresponds to the de Broglie hypothesis if linear momentum p is mapped into a spacetime three-vector

$$p = p^\alpha e_\alpha \to p^D = p^\alpha e_\alpha^D; \qquad e_\alpha^D = e_0e_1e_2e_3e_\alpha = \gamma_5 e_\alpha \tag{31b}$$

[which we called the *geometrical momentum* in Keller (1984, 1986*a*), which references are hereafter referred to as K1, K2]. The e_α^D are usually called axial vectors (a generalization to ST of the Gibbs construction).

The constant $h/2\pi$ introduced here is not really a property of the geometrical frame of reference, but a quantity relating the basic fields to the frame: each field is assigned a number of volume elements h which are to be identified with the possible *actions* of the field. The basic fields exist forever, either in their original form or transformed by the interactions (represented by a redefinition of the action below); then the total action of the universe will be infinite after an infinite time, but for a finite universe the total action per unit time is a finite quantity and only actions of hypervolume h are distinguishable as physical processes. Then if the boundary conditions fix an extensive quantity—let us say the fact that after a rotation by 2π we come back to the original situation—then the action has to be nh and a quantity $L_{ij} = nh/2\pi$ is well defined.

Our postulate III defines an arbitrary particular form to consider and use $\mathcal{D}_c$, which is unavoidable at this level of development of the theory; it fixes at the same time the splitting $\mathcal{D}_c = \mathcal{D} + i\mathcal{D}$ and it also requires that a coordinate parametrization of ST exists. This postulate relates the homogeneity and isotropy of the background universe to the properties of the fields that we study in it.

The finiteness of the physical universe will also carry an uncertainty in the definition of average quantities. For example, the definition of a *position*, already limited by the condition on the action, is also limited by the fact that the centroid of a group, as a physical origin, has a probability distribution because the different fields are in principle uncorrelated. Then the physical frame we are generating will carry uncertainties in the definition of quantities which are to be considered when the geometrical multivector frame is taken as a representation of the actual physical spacetime.

There are three types of statistical uncertainties to be considered; one is due to the finiteness of the number of fields N_0 considered, a second is due to the average number of fields n_0 generating a volume element; and a third is due to the (large-scale) local variations in that number of fields n generating a unit volume $n = n_0(1+\varepsilon)$. These statistical uncertainties are to be added to the physical uncertainty in spacetime position due to the finiteness of the quantity we have called the hypervolume element which is allocated to a physical action.

The notion of distance needs more elaboration; this will be done in Sections 4–6.

3.2. The Spacetime Geometry as Generated by a Uniform Background Electromagnetic Radiation

To show the way the geometry of spacetime can also be considered as generated by a uniform background of electromagnetic radiation, let us start our circuitous reasoning by *defining* that a four-component spinor f of the form

$$f = \begin{pmatrix} \mathbf{f}_a \\ \mathbf{f}_b \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \sum_\alpha f^\alpha \chi^M_\alpha, \qquad \text{with} \quad \mathbf{f}_a = \begin{pmatrix} f_3 \\ f_1 + if_2 \end{pmatrix}, \quad \mathbf{f}_b = \mathbf{f}^c_a = \begin{pmatrix} f_1 - if_2 \\ -f_3 \end{pmatrix} \tag{32}$$

represents a massless spin-1 field with the properties of the electromagnetic field. f is not a solution of the Dirac equation (DE), as its components are related in a way that will not be a solution of DE. They are related instead in such a way that they do not correspond to a left–right-handed pair of the same spin, but to twice a right- (left-) handed spinor corresponding to the same definite spin. Their relative phase is also different from the relative phase of the left–right-handed spinor of the Dirac electron theory.

When the correspondence with the electromagnetic field is made, the quantities $f_k = F_k$ are

$$f_k \equiv \frac{1}{\sqrt{2}} (\mathbb{H} + i\mathbb{E})_k, \qquad k = 1, 2, 3 \tag{33}$$

with $\mathbb{H}$ and $\mathbb{E}$ being, respectively, the magnetic and the electric field strength.

In the same form we define that a spinor corresponding to a spacetime vector x_u is

$$\psi = \begin{pmatrix} x_0 + x_3 \\ x_1 + ix_2 \\ x_1 - ix_2 \\ x_0 - x_3 \end{pmatrix} \tag{34}$$

and for example a current vector j is

$$j = \begin{pmatrix} \rho + j_3 \\ j_1 + ij_2 \\ j_1 - ij_2 \\ \rho - j_3 \end{pmatrix} \tag{35}$$

The spinors representing the electromagnetic field and the charge currents fields are more easily seen in their corresponding multivector form by applying the transformations

$$Sf = \mathcal{F} \qquad \text{and} \qquad Sj = \mathcal{J} \tag{36}$$

with

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & -i & i & 0\\ 1 & 0 & 0 & 1\end{pmatrix}, \qquad S^{+}S=1 \tag{37}$$

to give

$$\frac{1}{\sqrt{2}}\mathscr{F}=\begin{pmatrix}0\\ F_1\\ F_2\\ F_3\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\mathscr{J}=\begin{pmatrix}\rho\\ j_1\\ j_2\\ j_3\end{pmatrix} \tag{38}$$

obeying the spinorial equation

$$\alpha_{\mu}^{M}\partial^{\mu}\mathscr{F}=\mathscr{J} \tag{39}$$

with $\alpha_0^M=1$ and $\alpha_k^M=Si\gamma_5\gamma_0\gamma_k S^{\dagger}=iS\gamma_{ij}S^{\dagger}$, $i, j, k=1,2,3$ cyclic (the "source" J should not appear here, as we have not defined F as an interaction field yet). Here $e_{\mu}\rightarrow\gamma_{\mu}$ again.

These equations were first derived by Oppenheimer (1931), Ohmura (1956), and Moses (1958), considered by those authors to be the Maxwell equations in "Dirac form," and shown by Keller and Rodriguez (1984) to be a mapping of the multivector Maxwell equations. Note that they are not homogeneous or eigenvalue equations (there is no $\mathscr{J}\mathscr{F}$ term).

In this representation the basis vectors e_{μ}^{M} are

$$e_0^M=\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\end{pmatrix}, \qquad e_i^M=\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$$

$$e_2^M=\begin{pmatrix}-i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & i\end{pmatrix}, \qquad e_3^M=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & i & 0\end{pmatrix} \tag{40}$$

and the spin matrix

$$\Sigma_{12}^{M}=ie_1^M e_2^M=i\alpha_1^M\alpha_2^M$$

with eigenspinors $(1\,0\,0\pm 1)$ and $(0\,i\pm i\,0)$, that is, an inseparable Weyl spinor pair, both of the same chirality, insofar as the spin matrix

$$\mathbb{M}_s^M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \Sigma_{12}^M \tag{41}$$

mixes the two Weyl spinors; this is needed as a result of the space-free electromagnetic field being of a definite chirality.

The multivectors e_A^M are generated by the extended Cartan map also,

$$e_A^M = \sum_{\alpha\beta} e_A^{M\alpha\beta} \chi_\alpha^M \chi_\beta^{M+} \tag{42}$$

from the basis spinors χ_α^M. Then a radiation background defines a multivector spacetime geometry. And, tautomerically, the spacetime geometry defines the basis spinors χ_α^M as its left (right) modules.

Equation (39) could be written in terms of the original spinor j, instead of the transformed J, by a similitude transformation to it:

$$S^+ \alpha_\mu^M \partial^\mu \mathscr{F} = S^+ \mathscr{J} \to \alpha_\mu'^N \partial^\mu f = j \tag{43}$$

where the spinorial pair relationship is explicitly shown in the f and j. Here $\alpha_\mu'^{\mathbb{M}} = S^+ \alpha_\mu^{\mathbb{M}} S$.

4. THE EQUATION FOR THE MATTER FIELDS

4.1. Matter as Spinor Fields Describing Nondecaying Quanta in Spacetime

Once we have found that the geometrical relationships created by spinors correspond to the multivectors of spacetime, we have to come to the description of matter and interaction fields as those fields which can exist in the background universe. This will be done here and in Section 5, where, moreover, we have to show that the relations between the fields enlarges our local multivector geometries into the extended body we have called the "universe." This will be necessary because the motion of spacetime distance is not directly produced by the spinor to multivector mapping.

Postulate III above assumed that a parametrization resulting in the standard coordinates of spacetime was possible, and in fact gave a first contribution to fixing scales because it requires that the quantity $h = 2\pi(L_{ij}$ or $p \cdot x) = S$ should be obtained by combining characteristics of the spacetime and of the physical fields to be described in it through a position distribution and a moment distribution.

The first approximation will be then to consider a "free" field in the universe. The frame we have called the universe that has been created has vector dimension $N=4$ when the real Clifford algebra $\mathscr{D}$ is used, corresponding to ST. In this background a matter field d is required to have an *action* $S_d = nh$. Mathematically, this requires the use of an eigenvalue equation which should be invariant under the basic symmetries of the ST (which was postulated to be homogeneous and isotropic).

The spinors describing a matter field would be expressed in the spinor basis χ_a as

$$\psi_d(x) = \sum_a \psi_d^a(x)\chi_a \tag{44}$$

The basis operations in ST, rotations, generated by the operator Ω, and translations, generated by an operator D, should leave it unchanged; this requires that ψ_d should be a simultaneous eigenspinor of Ω and D:

$$\Omega\psi_d = l\psi_d \qquad \text{and} \qquad D\psi_d = p\psi_d \tag{45}$$

The first condition requires that a quantity l (which corresponds to angular momentum) should be associated with the field and the second that a second quantity p (proportional to the energy-momentum) should also be associated with the field.

The displacement generator operator has vector character in ST; then p is also a vector eigenvalue. The generators of the displacements are the operators $D = b\gamma_\mu\partial^\mu$ to be used below in Section 4.2, where we will show how Postulate III results in the de Broglie phase.

4.2. The Multivector Dirac Equation

We follow here our previous treatment (K2), making considerations more precise when needed for our present purposes.

An observer in reference system $\mathscr{S}$ (or $\mathscr{S}'$) associates an energy-momentum vector p (or p') to an electron (in fact, to any "elementary" particle of mass $m_0 \geq 0$ and "spin up")

$$p^\beta\gamma_\beta = p'^\alpha\gamma'_\alpha \tag{46}$$

where the basis vectors of $\mathscr{S}'$ and $\mathscr{S}$ are related through a Lorentz transformation

$$\gamma'_a = \mathscr{L}\gamma_a\mathscr{L}^{-1}; \qquad \mathscr{L}\mathscr{L}^{-1} = \mathscr{L}^{-1}\mathscr{L} = 1 \tag{47}$$

If the observer $\mathscr{S}'$ is taken to be that where $p' = m_0c\gamma'_0$, then postmultiplying (46) by $\mathscr{L}P_+P_\uparrow = \mathscr{L}P_{+\uparrow}$, we obtain an equation in multivector form relating the particle's system to the observer's system $\mathscr{S}$:

$$p^\beta\gamma_\beta\mathscr{L}P_{+\uparrow} = m_0c\mathscr{L}\gamma_0P_{+\uparrow} \tag{48}$$

Introduce (see Hestenes, 1966) the Schrödinger operator $\hat{p}^\beta$, proportional to the displacement generators,

$$\hat{p}^\beta \mathscr{L} P_{+\uparrow} \equiv \hbar \partial^\beta \mathscr{L} P_{+\uparrow} \mathbf{I} = p^\beta \mathscr{L} P_{+\uparrow}, \qquad \text{with} \quad \mathbf{I}^2 = -1 \tag{49}$$

to obtain the multivector Dirac equation (or Dirac–Hestenes equation)

$$-\gamma_\beta \partial^\beta \mathscr{L} P_{+\uparrow} = m_0 c \mathscr{L} P_{+\uparrow} \gamma_0 \mathbf{I} \tag{50}$$

where $h/2\pi = 1$ and $\mathbf{I}$ is some rotation plane. Hestenes (1966, 1975) proposes $\mathbf{I} = \gamma_{12}$, but for the analysis of the rest of this section we need $\mathbf{I}$ to commute with all bivectors and accordingly our only possibility is $\mathbf{I} = \gamma_5$ (given by Postulate III). As a consequence of our choice, the rest mass of a physical particle will not be a primitive concept, but the result of the interaction between left- and right-handed component fields (see below). The replacement $\mathscr{L} P_{+\uparrow} \to \Psi_0$ conceals the need of $P_{+\uparrow}$ in (50).

The general solution $\mathscr{L} = L Q_0$ to the multivector equation (50) where $Q_0 = A \exp[-\mathbf{I}\mathbf{p} \cdot \mathbf{x}/(h/2\pi)]$ can be "gauged"

$$\mathscr{L} P_{+\uparrow} \to \Psi = A' \exp\{-\mathbf{I}[\mathbf{p} \cdot \mathbf{x} + \phi(\mathbf{x})]/h\} \tag{51}$$

if the differential operator is generalized to a covariant derivative $\gamma_\mu \partial^\mu \to \gamma_\mu D^\mu$. In Section 5 we give a detailed discussion of D^μ. In (51) the more general gauge "angle" is a multivector corresponding to the even algebra of $R_{1,3}$, which then commutes with $\gamma_5 = \gamma_{0123}$,

$$\phi(\mathbf{x}) = \phi_{\text{scalar}}(\mathbf{x}) + \gamma_5 \phi_{PS}(\chi) + \gamma_\mu \gamma_\nu \partial^\mu \Omega^\nu(\mathbf{x}) \tag{52}$$

The scalar part is usually interpreted as corresponding exclusively to the electromagnetic field, but in our theory we interpret concerted contributions from the scalar and from the pseudoscalar parts, after a more general definition of D^μ (Section 5), as corresponding to the weak and color fields; finally, the bivector part contains the effect of the gravitational field (Keller 1984, 1986*a,b*). That is, the interaction fields are given as boundary data in the phase of Q to represent both the rest of the physical world and the physical effect of the particle on itself. The origin of the color and of the electroweak interaction symmetries between different fields will be discussed below (see also Keller and Rodríguez-Romo, 1990).

As usual, the electromagnetic interaction appears as a (complex) phase factor $e^{-\mathbf{I}\phi}$, which will produce an "extra" energy-momentum $\partial^\mu e^{-\mathbf{I}\phi}\mathbf{I} = (eA_\mu/c)e^{-\mathbf{I}\phi}$, the A_μ being the components of the usual electromagnetic field vector. The weak and color fields will also produce an "extra" energy-momentum of mixed vector–"axial" vector character and the gravitational field changes the local, fiducial, frame $\gamma_\mu \to \gamma''_\mu(x)$. The gravitational interaction arises because, in order to compensate such a gauge transformation, a vierbein is needed (K1),

$$f_\mu = (f^0\, e^{-\partial\Omega})_\mu = f^\alpha_\mu f^0_\alpha; \qquad g^0_{\alpha\beta} = f^0_\alpha \cdot f^0_\beta = \operatorname{diag}(1, -1, -1, -1) \tag{53}$$

where the f^0 are locally Lorentzian tetrads,

$$g_{\mu\nu} = g^0_{\alpha\beta} f^\alpha_\mu f^\beta_\nu = [g\, e^{-2\Box\Omega}]_{\mu\nu} \tag{54}$$

defining a (gauge-invariant) gravitational "field"

$$\Phi^\nu_\mu = \partial_\mu \Omega^\nu + \partial^\nu \Omega_\mu - \delta^\nu_\mu \partial^\alpha \Omega_\alpha \tag{55}$$

which will obey, for self-consistency, the "field" equation

$$\Box^2 \phi = 4G\pi(\mathbf{T} - \tfrac{1}{2} g T) \tag{56}$$

with **T** the energy-momentum stress tensor of the total sources. This is different from the case of the scalar and pseudoscalar parts, where only the effect of sources other than the particle under consideration should be considered. The reason for this difference is the fact that the ST geometry is generated by all the fields, whereas the "extra" energy-momentum of the electroweak and color interactions is not directly referred to the frame, but to the description of the particle as quasi-free in that frame.

The additional terms introduced by the bivector part in (52) consist of two types of contributions.

The first is a universal term due to the average curvature (and its rate of change) of our isotropic and homogeneous system; as constructed in the previous section, it is represented by a diagonal tensor $\partial^\mu \Omega^\mu$ proportional to the time coordinate and results in a term like the right-hand side of (50), because it contributes with a quantity proportional to γ_0 (remember that γ_0 is the operator linking right-handedness to left-handedness in our representation); then it contributes to the rest mass of all particles where the possibility of a combination of left- and right-handed fields is given. Its role is exactly the same as that of the Higgs fields discussed in Section 5.3. The curvature of the universe breaks the right- to left-handed symmetry which is taken as a basis for the construction of the theory in the following paragraphs. In a curved universe of sufficiently large curvature, there would be no need for a Higgs field in order to give mass to the matter fields in the standard model of elementary particles.

In a slightly different sense (part of) the rest mass of the particle is already the result of introducing a covariant derivative in the equations to account for the constant average curvature of the universe, and we can say that a contribution to rest mass is obtained as a price for locally considering the universe to be flat in our equations below and explicitly eliminating the bivector term in (52).

The second part of the contribution from the bivector terms in (52) represents the local transformation of the coordinate system resulting in the local gravitational interactions for the particles, which in general could be taken into account via the standard formulas of general relativity unless higher-order terms had to be considered with the set of equations (53)-(56), once the covariant derivatives, as introduced below, are used.

The standard form of the free-particle Dirac equation is obtained (Casanova, 1976) using a spinor u (composite in our theory) such that $\gamma_0 u = u$ and $\mathbf{I}u = iu$, to define $\Psi u = \psi$ and $\Psi\gamma_0\mathbf{I}u = -i\psi$; here ψ is now a particular spinor projected out of Ψ,

$$i\gamma_\mu \partial^\mu \psi = m_0 c\psi \qquad \text{or} \qquad D\psi = -im_0 c\psi, \quad D^2 = \partial_\mu \partial^\mu \tag{57}$$

In our theory the choice $\mathbf{I} = \gamma_5$ from Postulate III is the only one allowing the choice (52) of the gauging. But the usual column spinors cannot simultaneously be eigenspinors of γ_0 and of γ_5 (recall that $\gamma_0\gamma_5 = -\gamma_5\gamma_0$); then for rest-mass particles we need a more general spinor representing a collection of fields.

There are two main possibilities in our spinor system to represent a collection of spinor fields $\chi_\alpha^{(d)}$ needed for composite particles: (1) as a matrix U consisting of rows of $\chi_\alpha^{(d)}$ or (2) as a supercolumn Ξ of $\chi_\alpha^{(d)}$. Each of these possibilities has its own advantages. The practical use of U is that it can be operated on the right by the elements of γ_A (if U contains four columns in the 4×4 representation of $\mathcal{D}_c$). The use of the column Ξ is otherwise standard in most of the elementary particle literature.

The standard procedure of constructing a supercolumn spinor will then be used below (see also K2) to give an explicit formulation of $SU(2)\times U(1)$ electroweak interactions, $SU(3)_{\text{color}}$ chromodynamics, and a unified presentation of $SU_c(3)\times SU_w(2)\times U_y(1)$ in terms of the gauging (52) of the diracon fields. In this theory $\mathbf{I} = \gamma_5$ in equation (50) will be used, corresponding to a plane of $\mathcal{D}_c$ in (22).

4.3. Multivector Generalization of the Dirac Equation

For a massless particle, $D\psi_0 = 0$. The ψ_0 obeys the Klein–Gordon equation with general solution Φ:

$$\partial_\mu \partial^\mu \Phi = 0 \qquad [\text{or} \quad -\partial_\mu \partial^\mu \Phi = (m_0 c)^2 \Phi] \tag{58}$$

from which the Dirac solution is obtained using the Dirac operator $D_0 = \gamma_\mu \partial^\mu$ to project it out,

$$\psi_0 = D_0 \Phi \qquad [\text{or} \quad \psi_0 = (D_0 + m_0 ci)\Phi] \tag{59}$$

We have used multivectors to generalize (50) and to develop the theory of symmetry-constrained Dirac particles (Keller, 1982*a*,*b*, 1984, 1986*a*,*b*). For this purpose we generalize the Dirac construction to a differential operator D-valued in the (complex) multivector algebra $\mathcal{D}_c$:

$$DD^\dagger = D^\dagger D = \partial^\mu \partial_\mu \tag{60}$$

The Klein–Gordon equation operator ($c = h/2\pi = 1$)

$$(\partial^\mu \partial_\mu + m^2) = (D^\dagger + mi)(D - mi) \tag{61}$$

requires $-D^\dagger m + mD = 0$; that is, either {A: $D^\dagger = D$ allowing that the *rest* mass $m \neq 0$ (Dirac's)} or {B: any D obeying (61) if $m = 0$}.

Let us restrict ourselves to case B (massless particles) and a D where we change (one or) several of the vectors γ_μ into a more general element γ_A. A hint comes from the special role of $i\gamma_5$ in elementary particle physics and from the general solution Φ above. For this purpose we have defined a set d of coefficients $\{t^d_\mu\}$ for the construction of a diracon operator D_d,

$$D_0 = \gamma_\mu \partial^\mu \to D_d = \left\{\cos(n + t^d_\mu)\frac{\pi}{2} + i\gamma_5 \sin(n + t^d_\mu)\frac{\pi}{2}\right\}\gamma_\mu \partial^\mu \tag{62}$$

or

$$D_d = a_d(\mu)\gamma_\mu \partial^\mu = \partial^\mu_d \gamma_\mu; \qquad \partial^\mu_d \equiv a_d(\mu)\partial^\mu \tag{63}$$

With the choice of n and t^d_μ integers, we obtain a set of diracon massless fields with definite chirality $i\gamma_5\psi_d = \pm\psi_d$. In that case $a_d(\mu) = \pm 1$ or $\pm i\gamma_5$ provided we also restrict $t^d_\mu = 0$ or 1, in order not to mix different chiralities.

Each D_d is characterized by the family index n and the particle field type set $\{t^d_\mu\}$ occurring in $a_\mu(\mu)$. The solutions to the massless Klein–Gordon equation (58) projected for a particular diracon field (46) are explicitly given by the immediate integration of the symmetry-constrained Dirac equation (K1, K2),

$$D_d\psi_d = 0; \qquad D^\dagger_d = a^\dagger_d(\mu)\gamma_\mu\partial^\mu; \qquad \psi_d = D^\dagger_d\Phi \tag{64}$$

as

$$\psi^0_d = B\exp(\mathbf{I}p^\mu_d x_\mu); \qquad p^\mu_d \equiv a_d(\mu)p^\mu \tag{65}$$

with $\chi_\mu = \chi^\nu g_{\mu\nu}$.

Before proceeding further, we must first allow for the gauging of (64) and (65). The wave function can be gauged by a phase angle $\Phi_d(x)$ if the "free" particle operator (63) is extended to the covariant derivative

$$D_d = \left[\partial^\mu_d - \mathbf{I}\frac{e}{\hbar}A^\mu_d(\mathbf{x})\right]\gamma_\mu \tag{66}$$

to obtain the gauged solutions of the generalized Dirac equation

$$\psi(\mathbf{x}) = B\exp\{\mathbf{I}(p^\mu_d x_\mu + \phi_d(\mathbf{x}))\} \tag{67}$$

with

$$A_d^\mu(\mathbf{x}) = A^\mu_{d,\,\text{scalar}}(\mathbf{x}) + A^\mu_{d,\,\text{pseudoscalar}}(\mathbf{x})\, i\gamma_5 + A^\mu_{\alpha\beta,\,\text{tensor}}(\mathbf{x})\,\gamma^{\alpha\beta} \tag{68}$$

and the multivector phase angles, generalizing the de Broglie phase,

$$\phi_d(\mathbf{x}) = \phi_{d,\,\text{scalar}}(\mathbf{x}) + \phi_{d,\,\text{pseudoscalar}}(\mathbf{x})\, i\gamma_5 + \phi_{d,\,\alpha\beta,\,\text{tensor}}(\mathbf{x})\,\gamma^{\alpha\beta} \tag{69}$$

Because both the coefficients $a_d(\mu)$ in (65) and the multivector ϕ_d commute with $\mathbf{I} = \gamma_5$, in the following we will replace $\mathbf{I}$ by its eigenvalues $\pm i$.

These solutions can be better arranged in families corresponding to a given value of n, with left-handed chirality (for $t^d_\mu = 0, 1$), and a corresponding set of antifamilies, with the negative quantum numbers n and t^d_μ, with right-handed chirality, as shown in Table I. Here a special collection of diracon fields has been made which will be useful (see below) for assigning a symmetry and name in accordance with the usual $SU_c(3) \times SU_w(2) \times U_y(1)$ standard theory classification. The value of $n = -1$ is chosen as a convenient reference to make the identification of the fields as simple as possible.

The phase factors $\phi_d(x)$ in equation (67) will allow the "interaction" and a resulting "transformation" of each of the basic diracon fields (64) among themselves [self-mapping according to a $U(1)$ scheme or mappings of several fields grouped in sets, with $SU(2)$ or $SU(3)$ schemes]; it will result that a full understanding of any one of the diracon fields and their identification with observed elementary particles can only be obtained if all particles are considered together. If we study each family (defined by a value n) by itself, in a first approximation, and consider the right- and

Table I. Allowed Sets of Symmetry-Constrained Quantum Numbers $\{t^d_\mu\}$ and $\{t'^d_\mu \equiv t^d_\mu + n\}$ for Chiral Fields Corresponding to the Electron Family ($n = -1$), Satisfying the Generalized Dirac Equation $D_d\psi_d = 0$[a]

t_0	t_1	t_2	t_3	t'_0	t'_1	t'_2	t'_3	Charge	Isospin	Color	Symbol	Name
0	0	0	0	−1	−1	−1	−1	−1	−1	—	e^-	Electron
2	1	2	2	1	0	1	1	+2/3	1	r	u_r	Up quark
2	2	1	2	1	1	0	1	+2/3	1	b	u_b	
2	2	2	1	1	1	1	0	+2/3	1	g	u_g	
0	0	1	1	−1	−1	0	0	−1/3	0	r	d_r	Down quark
0	1	0	1	−1	0	−1	0	−1/3	0	b	d_b	
0	1	1	0	−1	0	0	−1	−1/3	0	g	d_g	
2	1	1	1	1	0	0	0	0	0	—	ν_e	Neutrino

[a]The quantum numbers n, t^d_μ, and operator D_d are defined in equations (62)–(64) in the text. The negative value of n corresponds to the choice of e^- as reference. The charges are given by the average value $(t'_1 + t'_2 + t'_3)/3t'_0$ as described by the explanation of (72) in the text. The isospin pairs are connected by a change in the t'_μ such that $|t^d_\mu - t^d_\mu| = (2, 1, 1, 1) \bmod 2$, and the color triplets by a change in the t'_μ such that $t^d_\mu - t^{d'}_\mu = t^{d'}_\nu - t^d_\nu$.

left-handed electron fields together, we obtain a Spin(8) scheme similar to that discussed by Smith (1985), where it is shown, after some parametrization, to provide a sound description of observed elementary particle fields. In the rest of this paper we will develop from (62), (64), and (66) a basic physical scheme of the actual observable particles.

The physical origin of isospin is then the existence of a set of relationships between the diracon fields d with coefficients $a_d(\mu)$ in (64). The grouping of N diracon fields $d=1,\ldots,N$ in a subset gives rise to the $SU(N)$ symmetry with fundamental representation (Hermitian) matrices $\hat{T}_a$, the fields members of the set transforming into each other through a gauge transformation, as shown explicitly in the following sections. This grouping will allow the introduction of an isospin form of the gauge fields $A^\mu \to A^{\mu a}\hat{T}_a$. Then the symmetries of the gauging of the phase ϕ in (52) are transformed into the symmetry of the set of diracons.

The group of symmetries of the coefficients $a_d(\mu)$ is then the group of the unified presentation allowed by the ST algebra, with subgroups corresponding to the standard model.

5. CHIRAL GEOMETRY THEORY OF ISOSPIN AND COLOR: LAGRANGIAN FORMULATION OF THE THEORY OF DIRACONS

5.1. Chiral Geometry Theory of Isospin and Color

A formal presentation of the dynamics of symmetry-constrained Dirac particles or diracons can be given in terms of a Lagrangian for the collection of particles. We will deduce this Lagrangian from the equations of the preceding section and show that it corresponds to the postulated standard formulation of grand unified theories, as described, for example, in Close (1979), Field (1979), Okun (1982), or Halzen and Martin (1984).

Table I is useful for an overall presentation of the different particles, but it does not show the main symmetries of each type of field in the clearest form. In order to do so, we will first discuss the spacetime symmetries of the gauged fields, shown in Table I, generated by the quantum numbers t_μ^d (and n).

One should keep in mind that the Lorentz transformations L preserve the multivector character; in each of the different terms, k-vectors are mapped by L into k-vectors, even if the "components" change in the usual way; then equations (64) are multivector form invariant. This can be used to explore their symmetries. The same is true under spatial rotations.

For the quarklike diracons a more symmetric formulation can be given if the spatial coordinates are transformed in such a way that a reference local direction of motion γ_ν of the particle is defined to be $\gamma_\nu = (\gamma_1+\gamma_2+\gamma_3)\sqrt{3}$ and the notation $t_\nu^D \equiv i\gamma_5\gamma_\mu$ is used in such a way that we can explicitly exhibit the vector-(imaginary) axial vector momentum admixture and show that it is a constant (independent of the "color" of the diracon field).

Let us write in detail the energy momentum multivector ρ of every diracon field d, including the different "colors" red (r), blue (b), or green (g) of the quarks, according to formula (65) and Table I:

$$
\begin{aligned}
&\text{electron } e: && p_e = p^0\gamma_0 + p^\nu(\gamma_1+\gamma_2+\gamma_3)/\sqrt{3} \\
& && p_{\bar{u}}^{r} = p^0\gamma_0 + p^\nu(\gamma_1^D+\gamma_2+\gamma_3)/\sqrt{3} \\
&\text{quark } \bar{u}: && p_{\bar{u}}^{b} = p^0\gamma_0 + p^\nu(\gamma_1+\gamma_2^D+\gamma_3)/\sqrt{3} \\
& && p_{\bar{u}}^{g} = p^0\gamma_0 + p^\nu(\gamma_1+\gamma_2+\gamma_3^D)/\sqrt{3} \\
& && p_d^{r} = p^0\gamma_0 + p^\nu(\gamma_1+\gamma_2^D+\gamma_3^D)/\sqrt{3} \\
&\text{quark d}: && p_d^{b} = p^0\gamma_0 + p^\nu(\gamma_1^D+\gamma_2+\gamma_3^D)/\sqrt{3} \\
& && p_d^{g} = p^0\gamma_0 + p^\nu(\gamma_1^D+\gamma_2^D+\gamma_3)/\sqrt{3} \\
&\text{neutrino } \nu: && p_\nu = p^0\gamma_0 + p^\nu(\gamma_1^D+\gamma_2^D+\gamma_3^D)/\sqrt{3}
\end{aligned}
\tag{70}
$$

Here p^ν is the three-momentum and p^0 is the energy. We can see that the energy-momentum vectors are all in different phases of the $p_\mu \to p_\mu^D$ rotations.

Let us now consider a gauge energy-momentum vector field $A^\mu\gamma_\mu$ added to the diracon fields with coupling constant proportional to Q_e, modifying the *vector* part of the momentum, with the energy-momentum components given in the same proportion to the time part and to the spatial parts (calling $\gamma_\perp$ a vector perpendicular to the direction of motion γ_ν). For the electron

$$\mathbf{p}' = (p^0 + Q_eA^0)\gamma_0 + (p^\nu + Q_eA^\nu)\gamma_\nu + Q_eA^\perp\gamma_\perp \tag{71}$$

has components

$$
\begin{aligned}
&\text{timelike} && \gamma_0\cdot p = p^0 + Q_eA^0 \\
&\text{spacelike parallel} && \gamma_\nu\cdot\mathbf{p}' = p^\nu + Q_eA^\nu \\
&\text{spacelike perpendicular} && \gamma_\perp\cdot\mathbf{p}' = Q_eA^\perp
\end{aligned}
\tag{72}
$$

All of them are scalar quantities.

However, for a ū quark (taking, for example, a red quark, the result being invariant with respect to color),

$$\gamma_\nu\cdot\gamma_\nu^{\bar{u}} = \frac{1}{\sqrt{3}}(\gamma_1+\gamma_2+\gamma_3)\cdot\frac{1}{\sqrt{3}}(\gamma_1^D+\gamma_2+\gamma_3) = \frac{2}{3}+\frac{1}{3}i\gamma_5 \tag{73}$$

the scalar components will be affected by a factor of $\frac{2}{3}$, and following the same procedure for a down quark, the scalar components will be affected by a factor of $\frac{1}{3}$, and for a neutrino the scalar components will be affected by a factor 0.

Then if we make the obvious definition that only the scalar (proportional to 1) part of the gauge field, treated on an equal basis for the electrons and for the quarks or the neutrino, is to be considered first as gauged by the field A, the "charges" have to be Q_e, $\frac{2}{3}Q_e$, $\frac{1}{3}Q_e$, and 0, respectively. The pseudoscalar (proportional to $i\gamma_5$) parts are to be treated on a different basis, and will be shown to correspond to the weak and color interactions.

In the full Lagrangian, to be introduced and discussed below, a first term equivalent to the standard Dirac matter-field Lagrangian

$$\mathscr{L}_m = i\bar{\psi}D_\mu\gamma^\mu\psi \tag{74}$$

is to be replaced by the corresponding expression for diracons:

$$\mathscr{L}_d = i\bar{\psi}\partial_\mu\gamma_d^\mu\psi \tag{75}$$

It is in this term of the Lagrangian where we have to introduce an electromagnetic [scalar part of ϕ in (52)] gauging with a coefficient e for the electron field, $2e/3$ for the (anti) up-quark field, $e/3$ for the down-quark field, and 0 for the neutrino field. Then in the gauge theory we are constructing, the charges for the $U(1)$ part of the gauge fields are the (postulated usually) integer, fractional, or zero values of the standard theory. In general our method will allow us to *develop* a gauge theory instead of postulating it as in the standard approaches. In this form we are showing the physical origin of the various couplings of the gauge fields, and the role played by $i\gamma_5$ in it, as a part of the symmetry-constrained Dirac particle theory.

For this purpose the A field discussed above will have to be enlarged and split into contributions, usually called B and W^3 in the literature, and new "charges" T^3 and Y are introduced with the standard notation

$$Q = T^3 + Y/2 \tag{76}$$

but the assignment of T^3 and Y to give our values of Q will be straightforward and its physical origin clear.

It is convenient to start with a rearrangement of the set of diracon fields in groups which will show an explicit $SU(2)\times SU(3)\subset \text{spin}(8)$ symmetry as shown in Table I.

To start, we explore the $SU(2)$ relations; for each given family we can see that the addition of a set of symmetry coefficients $\{W^-\} = (0, -1, -1, -1)$, modulus -2, to the first row produces the last row and their addition to any one of the first group of three up-quark fields produces one of the group of three down-quark fields. That is: the same chiral phase change that takes the neutrino field into a left electron field will change an up quark into a down quark. The reverse process proceeds in the corresponding way. The "neutral" interaction will arise from a change in the phase of one of the partner fields canceling that of the change of the other.

In the language of bilinear spinor operators described before, we could write all these processes in terms of spinors: if $\{\chi_\nu, \chi_u, \chi_d, \chi_e\} = \chi_a$ represent the neutrino, up-quark, down-quark, and electron fields, respectively, and their respective dual fields are $\{\chi_\nu^\dagger, \chi_u^\dagger, \chi_d^\dagger, \chi_e^\dagger\} = \chi_a^\dagger$, with the orthogonality condition $\chi_a^\dagger \psi_b = \delta_{ab}$, then the processes above can be described by

$$\hat{W}^- = w^-(\chi_e \chi_\nu^\dagger + \chi_d \chi_u^\dagger) \tag{77}$$

$$\hat{W}^+ = w^+(\chi_\nu \chi_e^\dagger + \chi_u \chi_d^\dagger) \tag{78}$$

and the neutral interaction (to be combined with the electromagnetic) is

$$\hat{W}^3 = w^3 \tfrac{1}{2}(\hat{W}^+ \hat{W}^- - \hat{W}^- \hat{W}^+) \tag{79}$$

provided that, in order to account for the spin $h/2\pi$ of the gauge fields, in all cases the spins of each spinor operator of the product are opposite, i.e., that the spin of the electron field created is opposite to that of the neutrino field annihilated, etc. Then these processes correspond to vector interactions with total spin one, equal to the change in spin of the field during the interaction.

What we will show below is the correspondence between the interaction fields and each product of an interaction operator, written here in a formal way. We should add at this stage that, besides spin, energy-momentum is being exchanged during the interaction; for example, a photon interacting with an electron with energy-momentum exchange q could be written

$$\hat{A} = \sum_p \hat{\chi}_{e(p+q, \mp s \pm 1)} \hat{\chi}^\dagger_{e(p, \mp s)} \tag{80}$$

stating that the electromagnetic interaction annihilates an electron of momentum p and spin component s and creates an electron of momentum $p+q$ and of opposite spin.

The *color* interaction will change one of the spacelike t_i^d indexes of the quarks from the value 1 to 0 and produce a value 1 for one of the other indexes (which was zero previously), or change the axial vector momentum of two of those indexes simultaneously to a total of the eight operations $\{1 \to 2, 1 \to 3, 2 \to 3, 2 \to 1, 3 \to 1, 3 \to 2, 11 \to 22, 22 \to 33\}$, corresponding to the $SU(3)$ color symmetry; we can also write these results in a formal operator way if we add a color subindex to the quark fields; then

$$\hat{G}_{ab} = \hat{\chi}_{q,a} \hat{\chi}^\dagger_{q,b} \tag{81}$$

will correspond to a gluonic interaction changing color b into color a.

All these interactions in our diracon fields and in our chiral phase language correspond to a change in the wave function

$$\psi_d = u \exp(p^d \cdot x + \phi_d^0) = u \exp(\phi_d) \tag{82}$$

with u a spinor and the de Broglie phases ϕ_d being the sum of the scalar and the pseudoscalar parts of the products of the vector x with the momenta given by equations (75). The de Broglie phases are gauged by the ϕ_d^0 which also contain scalar and pseudoscalar parts. For the leptons the de Broglie phases are

$$\phi_{\text{electron}} = p^{\mu} x_{\mu} + \phi_e^0 \tag{83}$$

$$\phi_{\text{neutrino}} = p^0 x_0 + i\gamma_5 p^k x_k + \phi_\nu^0, \qquad k = 1, 2, 3 \tag{84}$$

The spinor u for the electron can be left- or right-handed, whereas for the neutrino, in order to satisfy equation (64), only the left-handed field is possible.

In order to preserve rotational symmetry, for each one of the quarks we need to show explicitly the gauge phase $\phi_{q,a}^0$ ensuring that the overall de Broglie phase is space-symmetric. This requires a complicated vector notation. If a space index is k (with values 1, 2, 3), a reference space index is $r = 1, 2, 3$, and a color index is a or b (with values r, b, g), we have a set of three multivectors [vector + i axial vector, $i = (-1)^{1/2}$],

$$e_k^a = c_k^{ar} \gamma_r; \qquad c_k^{ar} = \cos \omega_{rk} [\cos(\pi t_r^a/2) + i\gamma_5 \sin(\pi t_r^a/2)] \tag{85}$$

for each color a of a given quark, direction k in space, and quantum number t^a in Table I, for reference space direction r, this reference space direction at an angle ω_{rk} with the observer's space coordinates k. This is a more general notation than that of equation (70), where, for simplicity, the particle was taken to move in a direction with all $\cos \omega_{rv} = 1/\sqrt{3}$. The c_k^{ar} are then the sum of a scalar and (i times) a pseudoscalar.

For the purpose of our formalism we need a duality-symmetric set of coefficients b_k^{ar} such that $c_k^{ar} + b_k^{ar} = \cos \omega_{rk}$, the ordinary cosine directors (no axial vector mixing).

In terms of the multivectors (85) the de Broglie phases for the quarks are

$$\text{up quark} \qquad \phi_{u,a} = p^0 x_0 + c_k^{ar} p^k x_r + b_k^{ar} \phi^k x_r + \phi_{u,a}^0 \tag{86}$$

$$\text{down quark} \qquad \phi_{d,b} = p^0 x_0 + c_k^{br} p^k x_r + b_k^{br} \phi^k x_r + \phi_{d,b}^0 \tag{87}$$

The constants c_k^{ar} are different for up quarks and for down quarks, corresponding to the t_d^a quantum numbers.

Now, the phase angles ϕ_d^0 can either change the scalar–pseudoscalar structure of the de Broglie phases or leave them with the same structure.

In the first case we have a change of the particle's nature (the resulting wave function will obey a different wave equation), and in the second case we have a type-conserving interaction. For this purpose we construct a Lagrangian which is invariant to the changes of the phase structure of the different $\phi_d = \rho_d^u \chi_\mu + \phi_d^0$ shown above. We do this here using matrix notation for isospin to conform to the usual expressions of the standard theory.

5.2. A Lagrangian Formulation of Electroweak-Color Interactions

For the presentation of the Lagrangian we have at least two options: either we put all eight (left-handed) fields together in a column isospin matrix and show the pair of fields connected by $SU_w(2)$ symmetries and the triad of fields corresponding to $SU(3)_{\text{color}}$ interactions, or construct directly the $SU(2)$ doublets and the $SU(3)$ triplets. The first possibility is the more physical one, although it requires a less familiar and more elaborate notation. The second one shows directly the $U_y(1)\times SU_w(2)\times SU_c(3)$ symmetry in its clearest form and conforms to most current papers; for this reason we will use it here, at the expense of writing more terms in the Lagrangian.

First we need to state clearly that for each term ϕ^B in the phase angles in (82) we need to add in the Lagrangian densities a term $-gB$ in the (covariant) derivative, as usual in gauge theory, and a kinetic energy term $-\frac{1}{4}F_B^{\mu\nu}F_{\mu\nu}^B$ for the gauge field B, again in the usual way. But what is new in our approach is that if the gauge angles in (82) change the scalar-pseudoscalar structure of ψ_d, then the d field has been transformed into a new field, say d'; then in the Lagrangian the covariant derivative term will acquire an index $a=1,\ldots,N^2-1$, indicating that it corresponds to an $SU(N)$ type-changing interaction field and it will appear in the covariant derivative and in the Lagrangian multiplied by an $SU(N)$ matrix T_a; then $-gB\to -gB^a T_a$. The representation of the T_a matrices required here is the isospin (or color) step up or step down form.

Let us illustrate this for the electron-neutrino left-handed pair. We start with the definition of the $SU(2)$ isospin pair and its kinetic energy Lagrangian density,

$$\hat{K} = \mathbf{1}\gamma_\mu\partial^\mu; \qquad L=\begin{pmatrix}\psi_{\nu_L}\\ \psi_{e_L}\end{pmatrix}; \qquad \mathscr{L}_{K,L} = i\bar{L}\mathbf{1}\gamma_\mu\partial_\mu L \tag{88a}$$

and, in order to make it gauge invariant, we transform the kinetic energy operator into the standard covariant derivative,

$$\hat{K}_{w,y} = \begin{pmatrix}\gamma_\mu(i\partial^\mu - gW_3^\mu - \frac{1}{2}g'B^\mu) & g\gamma_\mu W_+^\mu\\ g\gamma_\mu W_-^\mu & \gamma_\mu(i\partial^\mu + gW_3^\mu - \frac{1}{2}g'B^\mu)\end{pmatrix}$$
$$= \gamma_\mu(i\mathbf{1}\partial^\mu - gW_a^\mu T^a - g'YB^\mu) \tag{88b}$$

with $T^a = \frac{1}{2}\tau^a$, the τ^a being the Pauli matrices, and Y a charge matrix, to obtain the $SU(2)$ gauge-invariant Lagrangian density

$$\mathcal{L}_{K,L} = \bar{L}K_{w,y}L \tag{89}$$

The Lagrangian density (89) will have three contributions, $\bar{e}_L e_L - \bar{\nu}_L \nu_L$, $\bar{\nu}_L(W_+)\, e_L$, and $\bar{e}_L(W_-)\nu_L$, which have some special properties: the first term is a concerted scattering where the energy-momentum given to one of the leptons is withdrawn from the other; the second asserts that $(W_+)\, e_L$ behaves like a neutrino, and the third that $(W_-)\nu_L$ behaves like a (left-handed part of an) electron. In the diracon theory we can immediately keep the definition of L and its use as a subindex, but in principle it was not needed to state that the neutrino was left-handed, because it *must* be left-handed to obey its wave equation. The kinetic energy operator is slightly changed through the use of the substitution $\gamma_\mu \to \gamma_\mu^d = a^d(\mu)\gamma_\mu$ defined in (46) to read for free fields

$$\hat{K} = \begin{pmatrix} i\gamma_\mu^{(\nu)}\partial^\mu & 0 \\ 0 & i\gamma_\mu^{(e)}\partial^\mu \end{pmatrix} \tag{90}$$

before it is explicitly made gauge invariant, and the $SU(2)$ part in the diracon theory, corresponding to (88b), is

$$\hat{K}_w = \begin{pmatrix} \gamma_\mu^{(\nu)}(i\partial^\mu - gW_3^\mu) & -\gamma_\mu[a^{(\nu)}(\mu)p_f^\mu - a^{(e)}(\mu)p_i^\mu] \\ \gamma_\mu[a^{(e)}(\mu)p_f^\mu - a^{(\nu)}(\mu)p_i^\mu] & \gamma_\mu^{(e)}(i\partial^\mu + gW_3^\mu) \end{pmatrix} \tag{91}$$

Here again, in the notation of our formalism, we have the following concerted interactions: the "neutral" interaction, where the moment given to the electron field $(-gW_3^\mu\gamma_\mu)$ cancels that given to the neutrino field; the "positively charged" interaction $[a^{(\nu)}(\mu)p_f^\mu - a^{(e)}(\mu)p_i^\mu]$, where an electron of initial moment $p_i^\mu\gamma_\mu$ appears in the final state as a neutrino field with momentum $a^{(\nu)}(\mu)p_f^\mu\gamma_\mu$; and, finally, the reciprocal, "negatively" charged interaction, where the initial state is a neutrino and the final state is an electron. All this is obtained through changes in the vector–axial vector coefficients $a^d(\mu)\gamma_\mu$. The equivalence of (91) and (88) is immediate if we now apply both to the L wave function, which is an eigenfunction of $i\gamma_5$ [or equivalently of the $a^d(\mu)$]: $-a^d(\mu)L = L$.

We finally obtain, as expected, the equivalence

$$gW_+^\mu = (p_f^\mu - p_i^\mu)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad gW_-^\mu = (p_f^\mu - p_i^\mu)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{92}$$

A similar procedure transforms a set of three colors of a quark field among themselves. Again it is advantageous to write the representation of $SU(3)_{\text{color}}$ in the eight operators (step up, step down, and color neutral interactions) between the three pairs of colors r–b, r–g, and b–g.

It is very important to consider that in all interactions there is a "conservation" of our t_d numbers because both the energy momentum and the axial parts are conserved during the interaction; Keller (1984) illustrates these rules (see the schemes in Casalbuoni and Gato (1980)).

Our procedure has been the following: (1) write the phase angles of the de Broglie phases [equation (69)], (2) introduce a covariant derivative for each component of the gauge phase angles [equation (68)], and (3) write the concerted pairs (or trios) of particle fields in the form of isospin or color multiplets, with the corresponding electroweak and color charges.

The complete Lagrangian density is

$$\mathscr{L} = \mathscr{L}_l + \mathscr{L}_B + \mathscr{L}_W + \mathscr{L}_Q + \mathscr{L}_{\text{mass}} \tag{93}$$

with

$$\mathscr{L}_l = \bar{L}\hat{K}_W L + \bar{\psi}_{e_R}\gamma^\mu[i\partial_\mu - g' B_\mu]\psi_{e_R} + \bar{L}\gamma^\mu(-\tfrac{1}{2}g' B_\mu)L \tag{94}$$

$$\mathscr{L}_B = -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{95}$$

$$\mathscr{L}_W = -\tfrac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu} \tag{96}$$

$$\mathscr{L}_{\text{mass}} = -G_e(\bar{L}\phi_h\psi_{e_R} - \bar{\psi}_{e_R}\phi_h^\dagger L) + |(i\partial_\mu - g\mathbf{T}\cdot\mathbf{W} - g' Y B_\mu/2)\phi|^2 - V(\phi) \tag{97}$$

$$\mathscr{L}_q = \bar{q}\hat{K}_{W,Y,G}\,q - \tfrac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}^{\mu\nu} \tag{98}$$

where we have introduced the kinetic energies of the gauge fields, the Higgs fields discussed below, and the short-hand $\hat{K}_{W,Y,G}$ for the kinetic energy term of the quarks with weak, $U(1)$, and color interactions:

$$\hat{K}_{W,Y,G} = \gamma_\mu \begin{pmatrix} P^\mu_{ur} & P^\mu_{urb} & P^\mu_{urg} & P^\mu_{udr} & - & - \\ P^\mu_{ubr} & P^\mu_{ub} & P^\mu_{ubg} & - & P^\mu_{udb} & - \\ P^\mu_{ugr} & P^\mu_{ug} & P^\mu_{ug} & - & - & P^\mu_{udg} \\ P^\mu_{dur} & - & - & P^\mu_{dr} & P^\mu_{drb} & P^\mu_{drg} \\ - & P^\mu_{dub} & - & P^\mu_{dbr} & P^\mu_{db} & P^\mu_{dbg} \\ - & - & P^\mu_{dug} & P^\mu_{dgr} & P^\mu_{dgb} & P^\mu_{dg} \end{pmatrix} = \gamma_\mu \mathbf{P}^\mu_q \tag{99}$$

Here we have used, for the gauged momenta of color-a quark q,

$$P^\mu_{qa} = ia_{qa}(\mu)\partial^\mu - gT^3_q W^\mu_3 - \tfrac{1}{2}g' Y_q B^\mu - g_G G^\mu_{aa} \tag{100}$$

with the definition

$$Q_q = T^3_q + Y_q/2 \tag{101}$$

For the color interaction between like quarks of colors a and b with initial momenta p_i and final momenta p_f,

$$P^\mu_{qab} = a_{qa}(\mu)p^\mu_f - a_{qb}(\mu)p^\mu_i \tag{102}$$

And for the weak interaction between quarks of types q and q' corresponding to the same color a,

$$P^{\mu}_{qq'a} = a_{qa}(\mu)p^{\mu}_{f} - a_{q'a}(\mu)p^{\mu}_{i} \tag{103}$$

Other interactions, for example, simultaneous color and type change, are not included here, for simplicity, but this kind of q–q' scattering can occur as a higher-order process and could be represented here as a set of two more entries in the matrix (81).

Again, as in the case of leptons, the identification of the standard W^{μ}_{+}, W^{μ}_{-}, and G^{μ}_{ab} fields can be done once the result of operating with $a_{qa}(\mu)$ on the wave function ψ_{qa} is known.

The need of a colorless combination c of quarks in order to make $\mathscr{L}_{\text{quark}}$ rotation invariant imposes a bound state between quarks adding up to $c = \bar{a}a$ or $c = \mathrm{r}+\mathrm{b}+\mathrm{g}$, that is, either

$$a_{qa}(\mu)p^{\mu}_{q} + a_{\bar{q}\bar{a}}(\mu)p^{\mu}_{q} = \cos\phi^{\mu}p^{\mu} \tag{104}$$

for a meson state, or

$$a_{q_1r}(\mu)p^{\mu}_{1} + q_{q_2b}(\mu)p^{\mu}_{2} + a_{q_3q}(\mu)p^{\mu_3} = \cos\phi^{\mu}p^{\mu} \tag{105}$$

for a baryon, showing that the momenta of the component quarks are not independent at any time. The hadron momenta include the gluon momenta, which, in turn, as shown in (102), depend on the quark momenta, the situation being very complex because the gluon–gluon interaction is possible and has to be included, as discussed in quantum chromodynamics. The intensity of the gluon fields is then fixed by the requirements of colorless elementary composite particles, the hadrons, and because this intensity is given by the gauge field equations relating it to the sources; this in turn generates a distance parameter, the size of the hadron or equivalently the range of the gluon field, which ensures that the hadron can be considered colorless (Keller, 1984).

5.3. Particles with Rest Mass

There are then two types of elementary particles: the quanta of the lepton fields and the composite elementary particles, the hadrons, which are composite, but cannot be divided without rotational symmetry being violated.

Equations (102) and (103) show explicitly the role of chiral symmetry in generating color, charge, and weak charge.

To understand the structure of the $\mathscr{L}_{\text{mass}}$ Lagrangian, we must recall that the γ_{μ} anticommute with $i\gamma_5$; for this reason the gradient operator changes a right- (left)-handed field into a left- (right)-handed field.

Then we have as the only choice for the neutrino left-handed field

$$\partial^{\mu}_{(\nu)}\gamma_{\mu}\psi_{\nu_L}=0 \tag{106}$$

but for the electron field we have the more general possibility of relating the left- and right-handed fields,

$$\gamma_{\mu}\partial^{\mu}\psi_{e_L}=m\,e^{i\theta}\psi_{e_R} \quad \text{and} \quad \gamma_{\mu}\partial^{\mu}\psi_{e_R}=m\,e^{i\theta}\psi_{e_R} \tag{107}$$

Then the possibility of the simultaneous existence of both (free) left- and right-handed electron fields allows the introduction of a new (mass) parameter, thus breaking the $SU(2)$ symmetry between the electron and the neutrino.

The expectancy value of γ_0 is the overlap of left-handed and right-handed components, so $\psi^{+}\gamma_0\psi=\bar{\psi}\psi$ is proportional to the mass which the field can acquire. A common normalization is $\bar{\psi}\psi=2m$. This is clearly seen in the Weyl representation of the γ_{μ}.

We have here a new type of gauge freedom where a combination of left- and right-handed fields can be mapped into itself. In the case of bound particles (always the case of quarks), the distinction between left-handed and right-handed fields vanishes because of the presence of the interaction fields in the momentum operator; besides, the kinetic energy of the particle and of the gauge field (gluons, etc.) will have to be added to the center-of-mass (rest) energy of the composite particle (proton, meson, etc.).

For particles with larger phase angle difference between the right-handed and the left-handed parts, special care has to be taken to account for the noncommutability of $\gamma_{\mu\nu}$ and the γ_{μ}.

The Higgs mechanism has to be chosen to explain the masses of the W within the Glashow-Weinberg-Salam theory and to express (106) and (107) in (97) because the left- and right-handed fields are independent.

In order to proceed with the discussion of the correspondence with the standard $U(1)\times SU(2)\times SU(3)$ color-electroweak interaction (Greenberg, 1982; Fritzch and Minkowski, 1974; Georgi and Glashow, 1974; Georgi, 1975; Salam, 1968; Weinberg, 1967), we need to identify the scalar Higgs field. This is easier if we first write it in a formal, spin operator way. All the interactions above were required to simultaneously change the spin of the interacting particles, but we can also construct new interaction operators with the (opposite) requirement that the spin is conserved during the interaction,

$$\hat{H}=h^{ff'}\hat{\chi}_{f,s}\hat{\chi}^{\dagger}_{f',s} \tag{108}$$

Here the operator $\hat{H}$ corresponding to the effect of the Higgs field will change the initial field f into the final field f' with the same spin, but f and f' need not be the same. Then the operator $\hat{H}$ will carry a new isospin I, equal to $I_f + I_{f'}$ in the same way as the $\hat{W}$ operators above carried isospin or the $\hat{G}$ operators carried color. A neutrino-electron $\hat{H}$, for example, $\hat{H}_{e\bar{\nu}}$ and $\hat{H}_{\nu\bar{e}}$, will carry one unit of isospin; four of those scalar operators can be constructed corresponding to the four pairs $e\bar{\nu}$, $\nu\bar{e}$, $e\bar{e}$, $\nu\bar{\nu}$. The last two have zero isospin, whereas the previous ones have -1 or $+1$ isospin, respectively. This is the origin, within chiral geometry theory, of the isospin of the Higgs fields. Their expression in terms of our diracon field and their chiral phases is given by (108). It is obvious that $\hat{H}$ and $\hat{W}$ cannot be independent; $\hat{W}$ changes the isospin of $\hat{H}$, and $\hat{H}$ will in turn define a reference vacuum for $\hat{W}$.

This scalar field will present an asymmetry with respect to the chiral set [spin(8)] of left-handed lepton and quark fields, because the electron field can be both left- and right-handed (e_L and e_R) and two terms will contribute in this case. Then the uncharged scalar, zero isospin, field will break the isospin symmetry among the scalar fields, due to the interaction between e_L and e_R.

In the matrix notation above, $\chi_d^+ = (\ldots, \chi_d^+, \ldots)$ is orthogonal by construction to $\chi_{d'}$ if $d \neq d'$. But the existence of nondiagonal terms in the Lagrangian shows that they are not physically independent; their relationships are explicitly formulated in the theory.

The Lagrangian (93) should be extended to include antiparticles. The particle-antiparticle formulation could look somewhat different for the electron field and for the neutrino or quark fields. We obtain the electron wave function from the even sum $e_L + e_R$ and the positron from the odd sum $e_L - e_R$; we see that there is a difference of $\pi/2$ in the relative chiral phases determining the character of the fields.

For the neutrino the phase difference is the same, but is usually given explicitly in the wave equation; for example, the positive energy solution of the neutrino, $E = |\mathbf{p}|$, satisfies

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}\chi = -\chi \tag{109}$$

corresponding, then, to the left-handed field, and the antineutrino, $E = -|\mathbf{p}|$, satisfies

$$\boldsymbol{\sigma} \cdot (-\hat{\mathbf{p}})\chi = \chi \tag{110}$$

and corresponds to a right-handed field. In both cases, for the particle-antiparticle, the change in phase is $\pi/2$ in the chiral plane $\psi_R \wedge \psi_L$, and there is no basic difference.

5.4. Interaction Fields and Duality

We end this section with two remarks (K2) about the geometrical interpretation which has been generated.

We first recall the notion of parity inversion **P** or space conjugation, consisting in the reversal of all the *spacelike* vectors of a multivector A; from the anticommutativity of the γ_μ,

$$\mathbf{P}\gamma_0 \to \gamma_0, \qquad \mathbf{P}\gamma_k \to -\gamma_k = \gamma_0\gamma_k\gamma_0 \tag{111}$$

Then, in general,

$$\mathbf{P}A = \gamma_0 A \gamma_0 = A^P \tag{112}$$

and the notion of Hermitian conjugation gives, from (112) as product reversal plus parity inversion,

$$A^\dagger \equiv \gamma_0 \tilde{A} \gamma_0 \tag{113}$$

(in the algebra defined in Section 2 parity inversion will change γ_5 into $-\gamma_5$).

It is very important to understand the geometrical origin of the currents appearing in the Lagrangians (93). The normalization $\bar{\psi}\psi = \psi^+\gamma_0\psi = 2m$ introduced above is the magnitude of a vector $J_0' = \Psi\gamma_0\Psi$ according to (48)-(50). For the system where the particle is at rest it is just two times the rest mass m of the particle and, because the Lorentz transformations are isometries of the vector algebra, it is the value in any (observer's) system S. The components of that vector in S are $J_0' \cdot \gamma_\mu$, or (in units of $-e$)

$$\begin{aligned} J_0' \cdot \gamma_\mu &= -e(\tilde{\Psi}\gamma_0\Psi\gamma_\mu)_s = -e(\gamma_0\Psi^\dagger\gamma_0\gamma_0\Psi\gamma_\mu)_s \\ &= -e(\Psi^\dagger\gamma_0\gamma_\mu\Psi)_s \end{aligned} \tag{114}$$

where the subscript s stands for scalar part, and (113) has been used (Hestenes, 1966, p. 44); premultiplying (114) by the unit dual (row) spinor u^+ and postmultiplying it by u, we obtain the usual definition of the (conserved) current

$$j_\mu = -e\bar{\psi}\gamma_\mu\psi \tag{115}$$

used in the Lagrangians.

We have shown before that we can obtain a deeper geometrical insight if we analyze the example of the plane wave solutions F of the field intensities of the free Maxwell equations

$$\Box F = 0; \qquad F = f e^{\gamma_5 k\cdot x} \tag{116}$$

the bivector f and the wave vector k being constant and obeying

$$\mathbf{k}f = k_0 f; \qquad k\gamma_0 = k_0 + \mathbf{k} \tag{117}$$

k_0 being a scalar and $\mathbf{k}$ a space vector with the conditions $k_0 = \pm|\mathbf{k}|$, obtained by multiplying (117) by $k_0 + \mathbf{k}$. If we apply to (99) the parity operation, f is transformed into $f^{\mathbf{P}}$ obeying

$$k_0 f^{\mathbf{P}} = -\mathbf{k} f^{\mathbf{P}} \tag{118}$$

showing that f and f^{P} behave as photon fields with the same energy and opposite momenta. On writing $f = \mathbf{e} + \gamma_5 \mathbf{b}$, where $\mathbf{e}$ and $\mathbf{b}$ are space vectors, we have that equation (117) corresponds to $k_0 \mathbf{e} = \gamma_5 \mathbf{kb}$ and $k_0 \mathbf{b} = -\gamma_5 \mathbf{ke}$, showing that $\mathbf{k}$, $\mathbf{e}$, and $\mathbf{b}$ are mutually perpendicular, $\mathbf{e}^2 = \mathbf{b}^2$, and $\mathbf{e} \cdot \mathbf{b} = 0$. The factor $e^{\gamma_5 k \cdot x}$ in (116) shows that $\mathbf{e}$ and $\mathbf{b}$ are rotating into each other with a phase angle $\phi = k \cdot x$ and that in the free field f^{P} the rotation takes place in the opposite sense $\phi^{\mathrm{P}} = -\phi = -k \cdot x$, corresponding to the two possibilities of circularly polarized light. For the photon field $p = (h/2\pi)k$ is the linear momentum; then we find again that the de Broglie phase $\phi = p \cdot x/(h/2\pi)$ corresponds to a duality rotation. The electromagnetic field, as in fact all gauge fields in our theory, adjusts the phases of the duality rotations between accelerated charged particles when photons are emitted or absorbed or when a particle is to be described in reference to other charged particles.

In the case of the Maxwell equations in the presence of a source, equation (116) should read

$$\Box F = J + K \tag{119}$$

The four-gradient operator in (119) has vector character and F is a bivector; then their product contains in general a vector part J and a trivector part K (Hestenes, 1966). Vectors and trivectors are dual to each other in spacetime. We have used in the Lagrangians the vector current (115) or electric charge current. The trivector part K corresponds then to a magnetic charge current

$$K_\mu = g\bar{\psi}_m \gamma_\mu^D \psi_m, \qquad \gamma_\mu^D = i\gamma_5\gamma_\mu \tag{120}$$

with ψ_m the wave function of the (thus far hypothetical) magnetically charged field m, and g the coupling constant for magnetic charges. The constant g is not independent of e (Dirac, 1931); for fermions

$$eg = nhc/4\pi, \qquad n = 0, \pm 1, \ldots \tag{121}$$

(for an elementary monopole $g = e/2\alpha$, $\alpha = 2\pi e^2/hc$); then no new physical constant is involved.

In our theory the non-Abelian gauge fields are related to trivector currents of the type (120); then we expect that g plays an important role and that for those currents and their gauge fields the replacement $e \to e/2\alpha$ should be done. If only one universal coupling constant e is kept for the several gauge fields, a factor $(e/2\alpha)^n$ should be absorbed into their definition. This is a natural choice in our theory. The constants a and n would be given by the self-consistency procedure: $\psi \to J + K \to F$, $(W, G$, or $H) \to \psi$, that is, computing the fields, consider them as sources, compute gauge fields, and use them to find the fields again. For the calculation of

W, G, or H we should use the charges e_d or g_d of the transition states, as well as the appropriate transition currents and effective masses of the boson interaction fields.

This is, however, not always the actual procedure in elementary particle physics, where very often the fields are considered free, transition probabilities are computed, and the final fields are considered free again, that is, where event probabilities are the main concern (including the spatial distribution of the cross section). In this case we have to compute the coupling constants from e, α, and parameters like the rest mass of the electron m_e or related quantities. It is important to realize that rest masses are related to coupling constants other than the electromagnetic, and that until a theory can give the vacuum expectation value $\phi_0^2 = |\phi_n|^2$ of the Higgs field in terms of α, this parameter (or m_e) remains an independent basic parameter of the model.

We could, for example, fix the electron mass $m_e = e\phi_0$, defining the Higgs field amplitude in terms of e, and then use the argument that the W's and Z^0 are linked to a magnetic current, in the circuitous manner proposed by Akers (1987), to obtain $m_w = m_e/2(2\alpha)^3$ for the mass of the W intermediate vector boson [that is, $m_w = 82.1$ GeV to be compared to the experimental value $m_w = 81.8 \pm 1.5$ GeV (Particle Data Group, 1986)], that is, the masses of the W particles are proportional to the Higgs field amplitude (and they should be proportional to the change of t_d numbers involved in the transition), and for the coupling constant

$$G_w = \frac{(2/3)^2(2\alpha)^6}{2^{1/2}m_e^2} = 1.165 \times 10^{-5}\,\text{GeV}^{-2} \tag{122}$$

in terms of α and m_e [accepted value $G_w = 1.165 \times 10^{-5}\,\text{GeV}^{-2}$ (Particle Data Group, 1986)]. This point deserves further investigation.

6. DISCUSSION AND CONCLUDING REMARKS

6.1. Basic Purpose

The theory developed in the preceding sections had as a basic purpose to show that the spacetime geometry contains all the elements necessary to describe the elementary particles and their interaction fields and that the reverse argument can be used, that we have used spacetime as a frame of reference because it is the geometry generated by a large collection of interacting spinor fields.

The revolution in physics that led to the standard model (SM), where the building blocks are taken to be quarks and leptons and their interacting (gauge) fields, is reflected in the current titles of elementary particle physics textbooks (for example, Close, 1979; Field, 1979; Huang, 1982; Okun, 1982; or Halzen and Martin, 1984). Therefore we had to structure our discussion to show that the standard model was fully included in our theory, with no attempt made to go beyond it. We obtained nevertheless besides a complete representation of all structural aspects of the SM, a logical scheme for other concepts (mainly that of confinement, or, equivalently in our theory, that observed particles should be colorless).

Quantum mechanics was given a setting in spacetime, but it is not discussed in its principles beyond what can be deduced directly from Postulate III and the spacetime geometry. Postulate III itself is needed to give an objective meaning to the relationship between the matter fields and the frame of reference we have called the universe. Even if we had to include the "wave functions" for the fields and their "wave equations," we remained at the level of one-particle approximations and many basic concepts of quantum mechanics were not even mentioned.

Postulate III introduced Planck's constant in the process of referring a field to the background universe. In order to do that, a *geometrical* momentum in $P^D(x)$ was defined, which is nothing else than an energy-momentum density, with h defining a unit hypervolume in spacetime. The definition of the energy-momentum density was basic to construct a wave equation stating that P is one of the defining properties of the field, which is invariant under translations in spacetime. The form that is computed and used leads otherwise to a probabilistic type of analysis of quantum mechanics. In fact, $P^D(x)$ defines a local velocity $V(x)$, with components $V_\mu(x) = \gamma_\mu \cdot V(x)$, and a new type of question can be asked: which is the flux (per unit area and unit time) for the field at point x? The density of particles, for example, is given by a quantity $\rho = [V_0^0/V_0(x)]^2$. As a consequence of this and of the wave equations, a complete analogy could be developed with the theory of moving dislocations in a lattice where the gauge fields will appear as perturbations arising from the presence of the particle fields, but these considerations are outside the purpose of the present paper.

It was also important to show that parameters (like m_e) or coupling constants (like e) were introduced in order to define a field as "free" as possible, so they represent the connection of the studied field to the rest of the universe. The concepts of volume and of distance had the same origin: the need to relate a given field to the rest of the universe in a form that will allow the splitting of the components of the system in a rational way. We had nevertheless one particular case where that separation is not possible, the case of the quark fields. The reason is that from the several types of fields we have discussed so far only two (e and ν) could be observed as free fields: massless neutrinos or massive electrons. A second type, isolated quarks, will break rotational invariance. But if in this case we demand that we should work with a composite field where such a combination of particles of the quark type is made that the ensemble is no longer rotational symmetry-breaking, then this second type of field may also be observed as an elementary free *composite* field.

We arrive then at a new concept. Ordinary composite particles, such as atoms and nuclei, can be split when energy is available into smaller components, whereas the composite elementary particles cannot, even if enough energy were available, unless a quark and antiquark are simultaneously created to restore rotational symmetry. This new type of particle, which, of course, corresponds to hadrons, will require, in order to preserve Lorentz and rotational symmetry, that three quarks (or a quark–antiquark pair) be together as a minimum in a small volume of space where there should be some coherence among the three quarks. This gives rise to a new type of interaction where each quark is constantly related to the other two in such a way that no particular "color" can be singled out. To achieve this, we need to associate each quark to a number of quanta of a symmetry-constrained gauge field with the complementary colors, gluons, all together adding up to the hadron's mass.

This is in fact the origin, in our theory, of the concepts of confinement and of the "bag" size which have been so fruitful in quark physics (see, for example, Close, 1979, Chapter 18).

6.2. Some Final Considerations on the Relation Between Multivectors and Spinors

We have explicitly shown the mapping of Dirac spinors to multivectors and used the energy-momentum vector conservation and the Lorentz transformations to derive the Dirac equation, which in turn defines the Dirac spinors themselves. The use of a larger subset of the multivector algebra—vectors and trivectors (actually γ_μ and $i\gamma_5\gamma_\mu$; then complex spacetime algebra was in fact used)—allowed us to generalize the Dirac equation to generate isospin and color symmetries. The solutions of the new equations generated a new set of spinors. We could then concentrate our attention on the multivector-spinor relationship. For this purpose it is convenient to use the general spinor expansion, given, for example, by Crawford (1985), and discuss the structure of the theory from that starting point. Crawford writes for the spinor ψ

$$\psi = e^{-i\phi}(\Sigma + \pi\gamma_5 + J^\mu\gamma_\mu - iK^\mu\gamma_5\gamma_\mu + \tfrac{1}{2}\delta^{\mu\nu}\sigma_{\mu\nu})\eta = e^{-i\phi}R^A\gamma_A\eta \quad (123)$$

where the real numbers R^A are the scalar (Σ), pseudoscalar (π), bivector ($\delta_{\mu\nu}$), vector (J_μ), and trivector (K_μ) parts, respectively. ϕ is a phase angle and η an arbitrary spinor [we can choose, without losing generality, $\eta^+\eta = 1$;

above we have called it u and given it the name Casanova spinor projector; see Casanova (1976)]. He points out that the Pauli–Fierz identities provide nine algebraic equations among the 16 multivector components R^A of the normalized bispinor densities (here the γ_A should be Hermitian):

$$\begin{gathered}\sigma = \bar{\psi}\psi, \qquad \pi = -i\bar{\psi}\gamma_5\psi, \qquad \Sigma_{\mu\nu} = \bar{\psi}\bar{\sigma}_{\mu\nu}\psi \\ j_\mu = \bar{\psi}\gamma_\mu\psi, \qquad k_\mu = i\bar{\psi}\gamma_5\gamma_\mu\psi\end{gathered} \tag{124}$$

The Pauli–Fierz identities among the 16 densities ρ_i are

$$\begin{aligned} j_\mu j^\mu &= \sigma^2 + \pi^2 \\ k_\mu k^\mu &= -j_\mu j^\mu \\ j_\mu k^\mu &= 0 \\ \Sigma_{\mu\nu} &= (\sigma^2+\pi^2)^{-1}[\sigma\varepsilon_{\mu\nu\rho\tau}j^\rho k^\tau - \pi(j_\mu k_\nu - j_\nu k_\mu)] \end{aligned} \tag{125}$$

The normalization of ψ is

$$R^A = (1/4N)\rho^A \tag{126}$$

with

$$N^2 = \tfrac{1}{4}\bar{\eta}(\sigma + \gamma_5\pi + j^\mu\gamma_\mu)[1 - i(\sigma^2+\pi^2)^{-1}(\sigma - \gamma_5\pi)k_\mu\gamma_5\gamma^\mu]\eta \tag{127}$$

Then we should write (123) in the form

$$\psi = e^{-i\phi}(\Sigma + \pi\gamma_5 + J_\mu\gamma^\mu - iK_\mu\gamma_5\gamma^\mu + \tfrac{1}{2}S_{\mu\nu}\sigma^{\mu\nu})\eta/4N \tag{128}$$

to show the explicit relation between a spinor and the corresponding multivector Ψ. Also $\Psi = \psi\eta^+$ as a Cartan map.

In the spinor (123) we have then eight independent quantities. If ψ is to be a solution of a generalized Dirac equation (Section 4.3), the phase angle ϕ can be gauged by the sum of a scalar, a pseudoscalar, and a bivector and the J_μ part as well as the K_μ part are gauge invariant after replacement $J_\mu \to J_\mu - eA_\mu$ and $K_\mu \to K_\mu - gA'_\mu$. There are then eight different, linearly independent, forms of choosing the coefficients R^A in (128), corresponding to our eight diracon fields of a given family in Table I. Other families are related to the first by a phase angle in multivector space. A special plane in multivector space is the plane W of the *vector* part of the momentum and the *trivector* part of the momentum [equations (71)-(73)], where the ratio of the vector part to the total momentum was the charge Q_d of the particle.

More generally, we should use three planes $W_\mu = (\gamma_1 - i\gamma_5\gamma_1, \gamma_2 - i\gamma_5\gamma_2,$ and $\gamma_3 - i\gamma_5\gamma_3)$, which together will define the charge, the isospin, and the color in the form described above. The weak interaction between different families f will require the use of W particles with different relative phases in a direction orthogonal to all three W_μ planes; we could say that then we will have a set of planes W_μ^f "tilted" relative to one another by successive 90° angles in a "flavor" plane perpendicular to the W_μ^+ planes. This will generate the Cabibbo angles in electroweak interactions.

Equation (128) shows that the ordinary Dirac spinors are the projections of a multivector

$$\psi = e^{i\phi} M\eta \tag{129}$$

$$M = (\Sigma + \pi\gamma_5 + J_\mu\gamma^\mu - iK_\mu\gamma_5\gamma^\mu + \tfrac{1}{2}S_{\mu\nu}\sigma^{\mu\nu})/4N \tag{130}$$

The multivector M is the normalized product of a set of projection operators and a basic multivector m_d,

$$M = P_a P_b m_d / 4N \tag{131}$$

This provides the connection with the alternative definition of a spinor as given, for example, by Lounesto (1980). M obeys an equation equivalent to the Dirac equation and has been discussed in detail by Hestenes (1966, 1975) for the case of the electron and neutrino fields.

The mapping $\psi\psi^T$ results in another multivector

$$\psi\psi^T = M_\psi = e^{-2\phi} P_a P_b |m_d|^2/4N = A\, e^{-2\phi} P_a P_b \tag{132}$$

which also obeys an equation similar to the Dirac equation. The idempotency of P_a and P_b makes in fact (132) almost equivalent to (131).

6.3. Considerations about the Mass, Momentum, and Angular Momentum of a Fermion in the "Universe"

We have considered a uniform background, the "universe," and then we have singled out a particle, a fermion represented by a spinor field. The fermion is then characterized by an energy-momentum and an angular momentum (this is described by the corresponding Dirac equation) and when a collection of other particles were also considered, the relation between them prompted the introduction of the interaction gauge fields.

But the energy-momentum and the angular momentum of a particle and "its" gauge field are not independent. In fact, it has always being recognized that the gauge field contains energy E, momentum p, and angular momentum S (see, for example, Higbie, 1988). The gauge field only appears when a second, test particle is considered; virtual gauge bosons have then to be included in this case, photons, for example. The main problem has been the need for either a distribution function or a cutoff in the integrals where E, p, or S are evaluated from the electromagnetic field generated by the electronic charge e and magnetic moment μ. Feynman *et al.* (1964) describe how, for the mass m_0 a cutoff should be $a = 2r_0/3 = 2e^2/12\pi m_0 c^2$, or $\frac{2}{3}$ of the classical electron radius. Higbie finds the same radius for the angular momentum (a is then of the order of 10^{-5} of a Bohr radius atomic unit). The value of the angular momentum is $S = -\mu(e/m_0)$, equal to the spin angular momentum of the electron, and he simultaneously obtains the correct g_s factor $g_s = 2$.

It is also possible to show that the circulation of the matter field ψ representing the electron carries the unit $h/4\pi$ of angular momentum or, equivalently, of course, that the expectation value of $\gamma_{03}/2$ is $h/4\pi$.

The angular momentum S of the gauge field should contribute to the virtual photons; two particles have to be considered in that case at least. The analysis of the two-particle electromagnetic field leads in a first approximation to the familiar e^2/r behavior for the energy of the gauge field. It is important then to recognize that the entity called "particle" should be understood as the particle in spacetime and that it includes in an inseparable form all of its gauge fields and the properties associated with them. These considerations point to the possibility that the Higgs fields are in fact a collection, which correspond in special ways to the different particles, and that again they provide the possibility of describing a nonlinear problem as a set of coupled fields.

As far as distributions or cutoffs in this type of analysis, we should remember that in our theory a distribution has a natural place as the uncertainties that should be given to the definition of position, uncertainties arising from the possible finite size of the universe and from the unavoidable fluctuations of the actual local densities with respect to the average universe density, both adding up to preclude the definition of distances smaller than those of a fraction of the classical electron radius. This is an indication that a complete theory should encompass both the microcosmos and the macrocosmos. That this is widely recognized can be seen, for example, in the list of topics of the resource letter on cosmology and particle physics of Lindley *et al.* (1988).

APPENDIX A. VECTORS AND MULTIVECTORS

The multivectors are generated by the antisymmetric, Grassmann, outer product $\wedge$ of a basis set $\{\gamma_\mu\}$ in N dimensions,

$$\gamma_{\mu\nu} = \gamma_\mu \wedge \gamma_\nu = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{A.1}$$

defining a two-vector and by recurrence the k-vectors; if the basis set is orthogonal, then a basis k-vector is

$$\gamma_A = \gamma_{\mu\nu\cdots\lambda} \qquad \text{(with } k \text{ ordered by place subindices)} \tag{A.2}$$

antisymmetric in all adjacent pairs of indices, and their involutions are the vector direction reversal

$$\gamma_A = (-\gamma_\mu)(-\gamma_\nu) \cdots (-\gamma_\lambda) \tag{A.3a}$$

and the products order reversal

$$\tilde{\gamma}_A = \gamma_{\lambda \cdots \nu\mu} \tag{A.3b}$$

The corresponding Clifford algebra is constructed using the Grassmann algebra and a symmetric inner (dot) product,

$$\gamma_\mu \cdot \gamma_\nu = \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) \tag{A.4}$$

to define the total, or geometric, product:

$$\gamma_\mu \gamma_\nu = \gamma_\mu \cdot \gamma_\nu + \gamma_\mu \wedge \gamma_\nu \tag{A.5}$$

The geometric product can be defined on a pair of general multivectors $\mathbb{M} = \sum_A M^A \gamma_A$ and $\mathbb{M}' = \sum_B M'^B \gamma_B$ by the rule that

$$\gamma_A \gamma_B = (\gamma_A \gamma_B)_{\mathrm{S}} + (\gamma_A \gamma_B)_{\mathrm{AS}} = \gamma_A \cdot \gamma_B + \gamma_A \wedge \gamma_B$$

where S means symmetric pairwise product part and AS means antisymmetric pairwise product part.

The metric of spacetime $R^{1,3}$ (where the indices $\mu = 0, 1, 2, 3$) is defined through the inner product

$$g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu = \mathrm{diag}(1, -1, -1, -1) \tag{A.6}$$

in such a way that the basis set consists of mutually anticommuting elements γ_μ.

If the multivector algebra C^N is considered as the complexification of $R^{m,n}$ $(N = m + n)$, we require the concept of absolute value square $|M_A|^2 = M_A \cdot M_A^K$ (which is not restricted to positive values), where M_A^K is a multivector with all coefficients being the conjugate of those of M_A. We can write formally for the complexification $\mathcal{D}_c \sim C^4$ of the spacetime algebra $\mathcal{D}$

$$\mathcal{D}_c = \mathcal{D} + i\mathcal{D} = \mathcal{D}_R + \mathcal{D}_I; \qquad \mathcal{D}_R^* = \mathcal{D}_R, \qquad \mathcal{D}_I^* = -\mathcal{D}_I \tag{A.7}$$

All multivectors are operators on themselves and on their spinors. The best-known examples of operations for $\mathcal{D}_c$ are: γ_0, generating the parity inversion P; γ_{123}, the time inversion T; γ_{0i}, the Lorentz boosts L; γ_{ij}, the space rotations R; $\gamma_5 = \gamma_{0123}$, the duality transformation D; and $i\gamma_5$, the chirality projection.

The pseudoscalar unit is $\gamma_5 = \gamma_{\mu\nu\lambda\rho}\varepsilon^{\mu\nu\lambda\rho}/4!$ in spacetime $R^{1,3}$, but it is simply i $[=(-1)^{1/2}]$ in $R^{0,5}$.

$\mathcal{D}_c$ can be regarded both as the complexification of the spacetime multivector algebra or as a five-dimensional space whose even subalgebra corresponds to spacetime, as shown in the main text.

A.1. Some Mathematical Properties of $\mathscr{D}_c$ Spinors

A.1.1. Classification of $L_{\mathscr{D}_c}$ Spinors

For complex spacetime the multivector $i\gamma_5$ plays a central role in the algebra; for this reason it is customary to define the main projectors Q_R and Q_L

$$Q_R = \tfrac{1}{2}(1 + i\gamma_5); \qquad Q_L = \tfrac{1}{2}(1 - i\gamma_5)$$

and name the two spinor subspaces generated by the Q on the spinor space $L_{\mathscr{D}_c}$ left-handed L and right-handed R, such that

$$L_{\mathscr{D}_c} = \mathfrak{L}_R + \mathfrak{L}_L \tag{A.8}$$

For $\mathfrak{D}_c$ = complex spacetime with dimension $N = 5$, the number of basic spinors is 2^p, with p = integer part of $(N/2) = 2$; then we need two projection operators A, which will render either the chiral representation $A = (i\gamma_5$ and $i\gamma_{12})$ or the standard representation (used in physics when massive particles have already been defined) with $A = (\gamma_0$ and $i\gamma_{12})$. The spinors will carry $n = 2^p$ indexes, either as a p-fold index or, as customary, a single index α taking 2^p values $(\alpha = 1, \ldots, 2^p)$.

If $L_{\mathscr{D}_c}$ spinors are acted by γ_μ or $\gamma_{\mu\nu\rho}$, they are mapped into spinors of the opposite chirality.

A.2. Covariant Vector and Spinor Derivatives

Following Hestenes (1966), define a differential operator $\Box$ by a series of mappings:

$$\Box_i \phi = \partial_i \phi \tag{A.9}$$

where ϕ is a scalar and $\Box_i$ maps scalars into scalars. For a vector field with basis R vectors γ_j,

$$\Box_i \gamma_j = -L_{ij}^k \gamma_k \tag{A.10}$$

and for multivectors A and B,

$$\Box_i(AB) = (\Box_i A)B + A\Box_i B \tag{A.11}$$

$$\Box_i(A + B) = \Box_i A + \Box_i B \tag{A.12}$$

In general, if $a = \sum_j a_j \gamma^j$, then

$$\Box_i a = (\partial_i a_j + a_k L_{ij}^k)\gamma^j \tag{A.13}$$

Hestenes (1966) uses this operator to discuss problems in general relativity [see also Hestenes and Sobczyk (1984) in this context].

For our spinor spaces $\mathfrak{L}$ and $\mathfrak{L}^+$ we can define

$$\Box_i \chi^\alpha = K_{i\beta}^{\alpha} \chi^\beta, \qquad \Box_i \chi^{\alpha +} = -K_{i\beta}^{\alpha} \chi^{\beta +} \tag{A.14}$$

where the $K_{i\delta}$ are related to the L^k_{ij} using the (representation-dependent) expansion of the $\gamma^k = \sum_{\alpha\beta} M^k_{\alpha\beta}\chi^\alpha\chi^{\beta+}$.

APPENDIX B. AN INVOLUTION OF COMPLEX SPACETIME GENERATED BY LIGHTLIKE SPINORS

B.1. Lightlike Spinors

We have defined a lightlike spinor as a pair ϕ of equal two-component spinors (η_1 and η_2; $\eta_1 = \eta_2$) from the same representation and the same spin and phase. The definition can be seen more clearly if we remember that a Dirac spinor (bispinor) corresponds to the set of one two-component spinor of one representation and one two-component spinor of the conjugated representation [as discussed, for example, in Landau and Lifshitz (1965)]. The lightlike spinor pair ϕ carries a total spin $s = 1$.

The mapping $\phi \to \phi\phi^\dagger$ generates the even part P_c of the spacetime multivector algebra: $P_c = \{1, \alpha^M_\mu, \alpha^M_{\mu\nu}, i1\}$ and is isomorphic to the set $\varepsilon P_c = \{\varepsilon, \varepsilon\alpha^M_\mu, \varepsilon\alpha^M_{\mu\nu}, i\varepsilon 1\}$ with ε an exchange operator interchanging $\eta_1 \leftrightarrow \eta_2$: The condition $\eta_1 = \eta_2$ should be built in, restricting the elements in the mapping. In the simplest matrix representation

$$\alpha^M_0 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}; \qquad \alpha^M_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{B.1}$$

with e the 2×2 unit matrix (sometimes denoted σ_0) and σ_i the Pauli matrices and

$$\varepsilon = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix} \tag{B.2}$$

In relation to the spacetime algebra represented by the γ_A, we can write

$$\alpha^M_j = i\gamma_5\gamma_0\gamma_j; \qquad \text{then} \quad \alpha^M_{123} = i1 \tag{B.3}$$

This could be somewhat physically misleading unless we consider that the role of $i\gamma_5$ is to ensure that chirality is a well-defined quantity for the lightlike fields and that γ_0 ensures that the pair (η_1, η_2) is considered as a unit.

The mapping presented in the text, to obtain the representation with the vector character of the matter current and the bivector (tensor) character of the electrical and magnetic fields, was given for theory-building reasons, but the electromagnetic interaction would be more apparent if the spinor pair formalism were used and the interaction diagrams written and interpreted as a conservation of spinor properties (spin and relative phases of the bispinors).

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